Solving D.E.'s Using Different Technologies

By Andrew Kasunich & Steven Turner

The motivation of this project was inspired by the array of different technical and mathematical courses we are taking this spring. Combined, Andrew and I are taking Differential Equations, Engineering Circuits, and MATLAB. The goal of this project was to combine and unify what we have learned in all three courses and apply it an analytical manner. For our project we chose to design and build a simple analog computer in order to solve the differential equation:

$$y'' + y = \sin(t), \text{ with initial conditions: } y(0) = 0 \ \& \ y'(0) = 0$$

What makes this project unique is we were able to solve the differential equation by applying techniques from MATH 32, ENGR 12, and ENGR 17. First we solved the D.E. analytically using Laplace transforms and plotted the solution using FreeMat. We then solved the D.E. in FreeMat by writing a simple function and program file that plotted our D.E. solution. Finally we designed a simple analogue circuit that solves the D.E. using operational amplifiers. A schematic and simulation was first made in Circuit Lab followed by construction/ debugging of our circuit on a breadboard. A function generator was used to input a sine curve into our circuit and using an oscilloscope we could see the output solution (Or at least something close haha). The following is a breakdown of our results.

1) Math 32: Using Laplace Transforms.

Our first goal, in this multi-step process, was to first find and solve a second order D.E. Using Laplace transform techniques learned in MATH 32, we solved by hand the equation $y'' + y = \sin(t)$ with the initial conditions: $y(0) = 0 \ \& \ y'(0) = 0$. Below is the solution to the equation.

$$y'' + y = \sin(t); \ y(0) = 0, \ y'(0) = 0$$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2 + 1}$$

$$\rightarrow Y(s) * (s^2 + 1) = \frac{1}{s^2 + 1} \xrightarrow{yields} Y(s) = \frac{1}{s^2 + 1} \cdot \frac{1}{s^2 + 1}$$

$$\rightarrow \pounds^{-1}\{Y(s)\} = \sin(t) * \sin(t) \xrightarrow{convolution} \int_0^t \sin(t - v) \cdot \sin(v) \, dv$$

$$\frac{1}{2} \int_0^t [\cos(t - 2v) - \cos(t)] dv = \frac{1}{2} \cdot \frac{\sin(t - 2v)}{-2} \Big]_0^t - \frac{1}{2} \cdot \frac{\cos(t) \, v}{} \Big]_0^t$$

$$= \frac{1}{2} \cdot \frac{\sin(t - 2t)}{-2} - \frac{1}{2} \cdot \frac{\sin(t)}{-2} - \frac{1}{2} t \cos(t)$$

$$= \frac{\sin(t)}{4} + \frac{\sin(t)}{4} - \frac{1}{2}t\cos(t) = \frac{1}{2}(\sin(t) - t\cos(t))$$

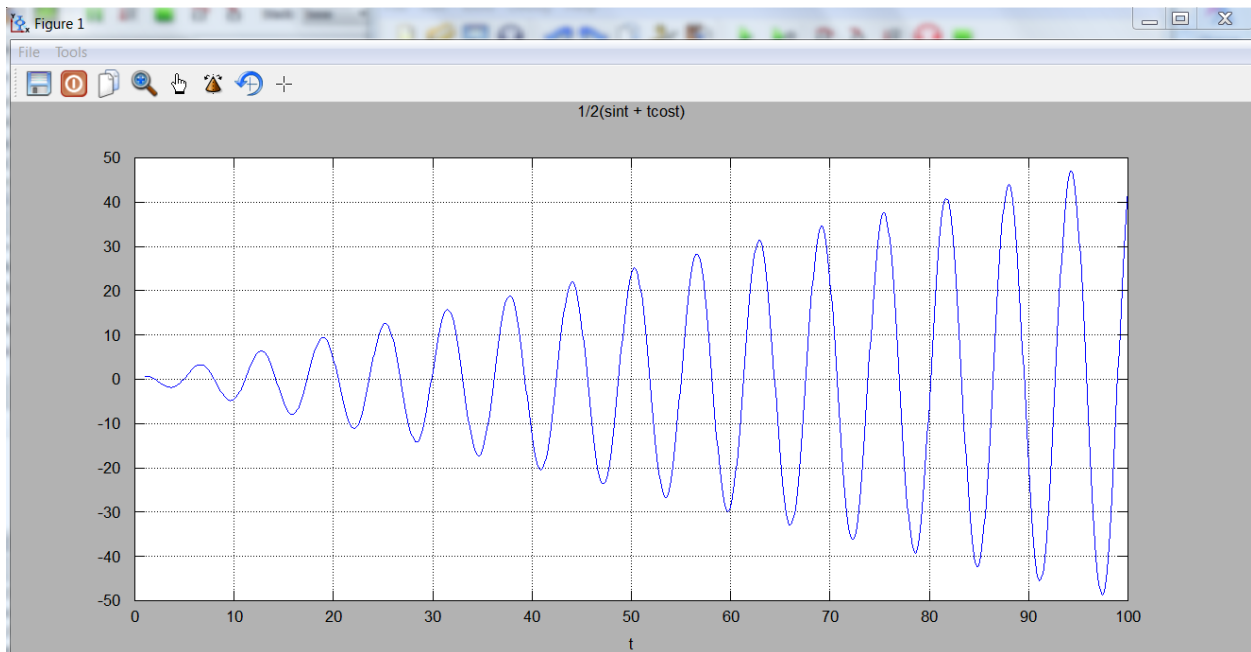*Figure 1 shows the hand solved solution plotted in FreeMat :*

Figure 1

Notice the t bearing cosine term in our solution. As t increases so does the amplitude of our solution.

2) ENGR 17 Technique: Programming in FreeMat

Below is the function and m file used in FreeMat. By turning a higher order differential equation into a linear system of first order equations we were able to use ode45 to give us the solution to our D.E. (*see figure 2 for the expected solution using ode45*).
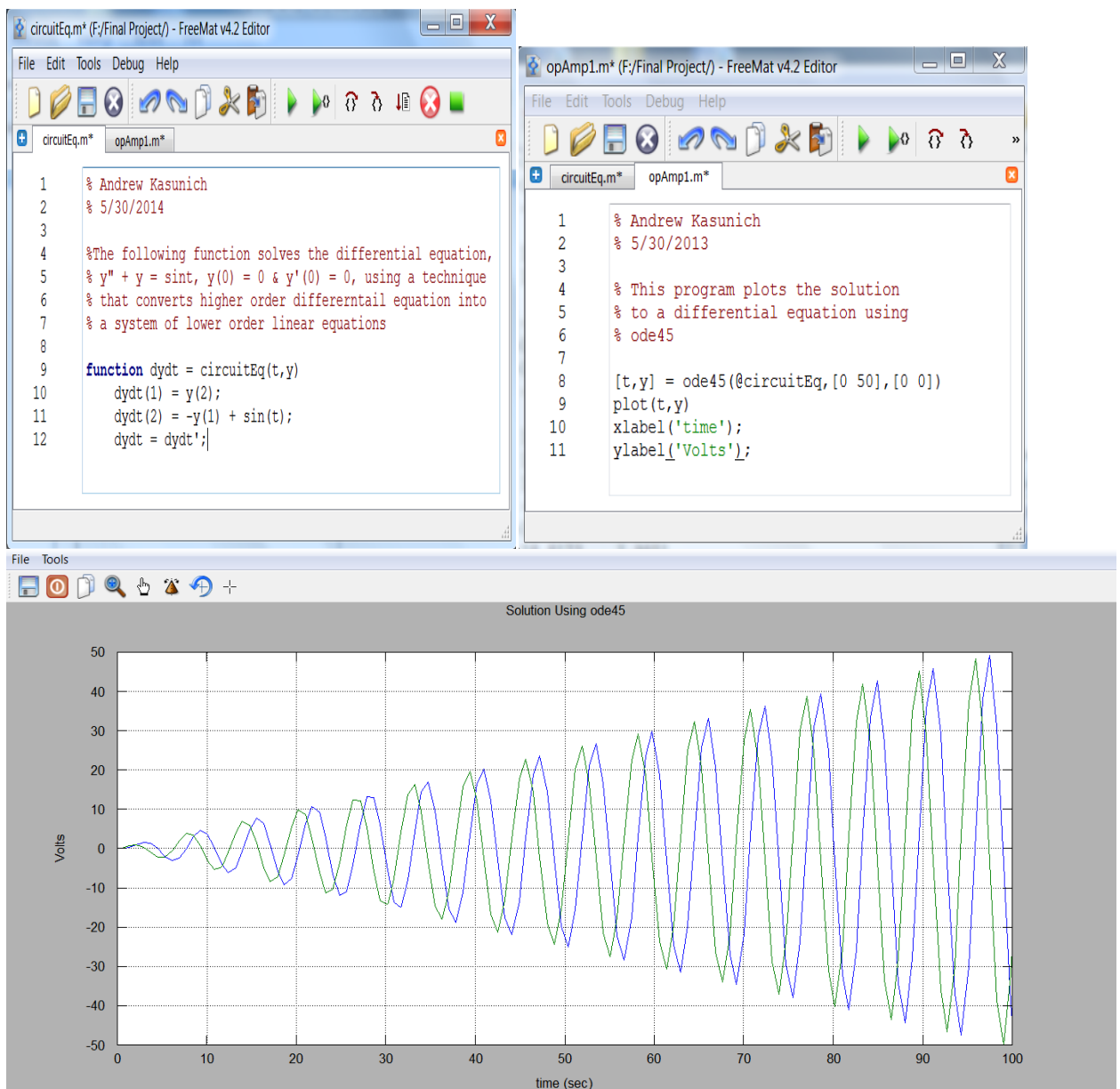


```
circuitEq.m* (F:/Final Project/) - FreeMat v4.2 Editor

File  Edit  Tools  Debug  Help

circuitEq.m*    opAmp1.m*

1    % Andrew Kasunich
2    % 5/30/2014
3
4    %The following function solves the differential equation,
5    % y" + y = sint, y(0) = 0 & y'(0) = 0, using a technique
6    % that converts higher order differerntail equation into
7    % a system of lower order linear equations
8
9    function dydt = circuitEq(t,y)
10       dydt(1) = y(2);
11       dydt(2) = -y(1) + sin(t);
12       dydt = dydt';
```

```
opAmp1.m* (F:/Final Project/) - FreeMat v4.2 Editor

File  Edit  Tools  Debug  Help

circuitEq.m*    opAmp1.m*

1    % Andrew Kasunich
2    % 5/30/2013
3
4    % This program plots the solution
5    % to a differential equation using
6    % ode45
7
8    [t,y] = ode45(@circuitEq,[0 50],[0 0])
9    plot(t,y)
10   xlabel('time');
11   ylabel('Volts');
```

Solution Using ode45

Figure 2

3) ENGR 12 Technique: Using Circuit lab & design/building a circuit with the breadboard.

Once the initial planning and brainstorming stage was complete we moved on to the next phase of our project; designing the circuit using tools from Circuit Lab. In order to complete this task Andrew & I first drew a rough abstract of what we needed on paper, as shown in figure 3. Next once the paper copy was completed we moved on to creating a virtual schematic of the circuit using Circuit Lab as seen in figure 4. In the final part of the designing phase we used Circuit Lab to plot a graph of both V_{in} and V_{out} functions as shown in figure 5. As was expected, our graphical plot in Circuit Lab matched the plot in FreeMat, so assuming all was well we moved on to the final phase of our project.
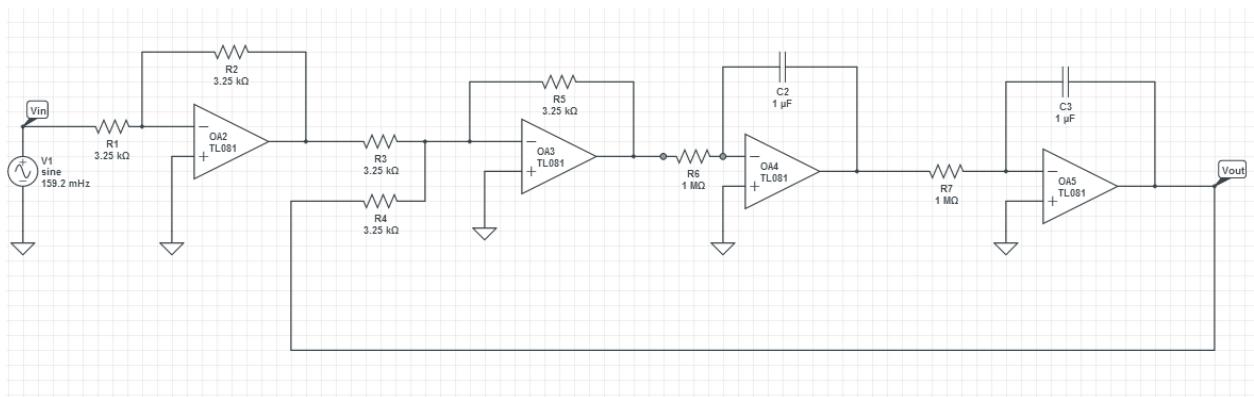
Abstract for $y'' + y = f(t)$
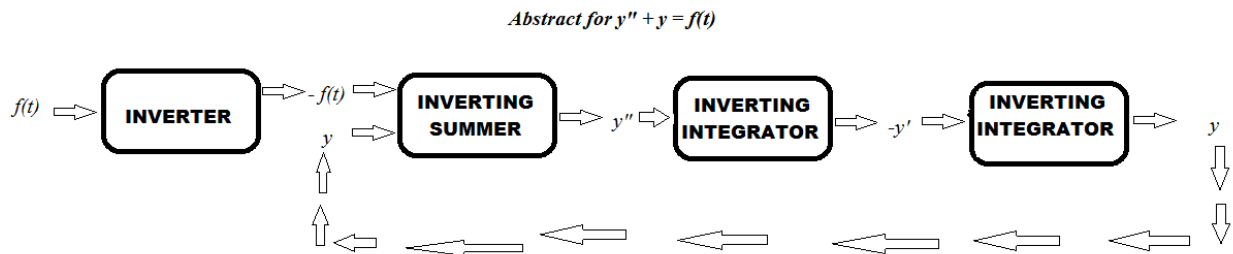
Figure 3

Figure 4 (Circuit Lab Schematic with Ideal Components)

Why Op Amps?

Operational amplifiers were the most important component in our circuit because they provided the integration needed to solve our differential equation. The beauty of the op amp is that they sort of act like infinite resistance. Thus the current is forced to flow through the capacitor. We have learned that the current flowing across the capacitor is, $i = Cdv/dt$, thus the voltage across the capacitor is 1/C multiplied by the integral of the current with the current equalling Input voltage divded by R. The output voltage is equivalent to the voltage

across the capacitor. Essentially an input voltage is turned into a current, and then integrated to yield our output.

$$Vin => i = Vin/R => 1/RC * \int Vin \, dt = 1/RC * Vcapacitor = 1/RC*Vout$$

If RC is made equal to 1, by setting R = 1MΩ and C = 1µF, then our scale factor is made equal to 1 and a clean solution should result.
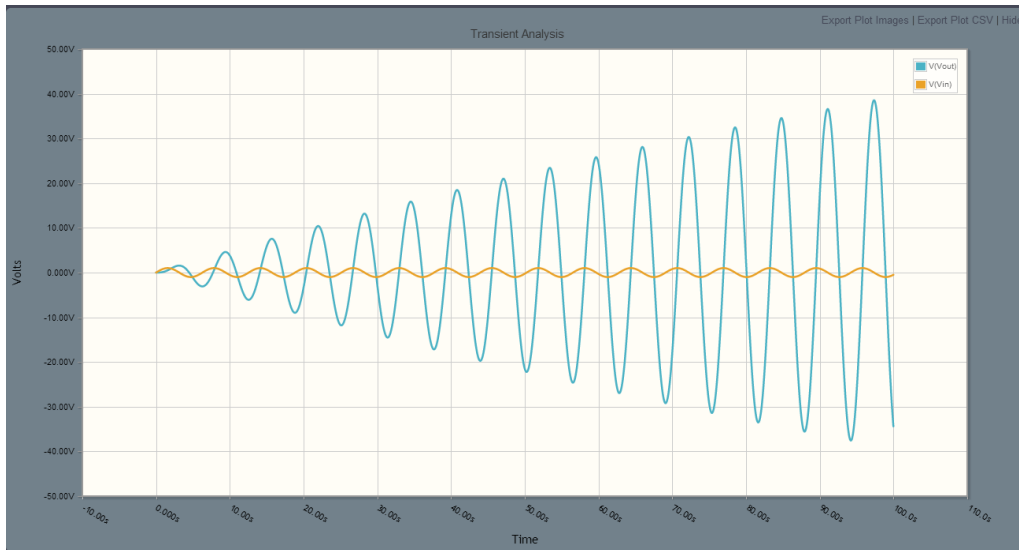

Figure 5 (Circuit Lab Plot with Ideal Resistors)

Our final phase of the project involved the actual building and wiring of the circuit. For this phase Andrew rigorously followed the schematic I had drawn up in Circuit Lab and was able to build the circuit shown in figure 6. Using an Ohm-meter, Andrew measured the actual values for the resistors we would be using and adjusted the Circuit Lab schematic accordingly. Figure 7 shows the circuit with the real measured values of resistance along with an adjusted graph to reflect the changes as shown in figure 8.
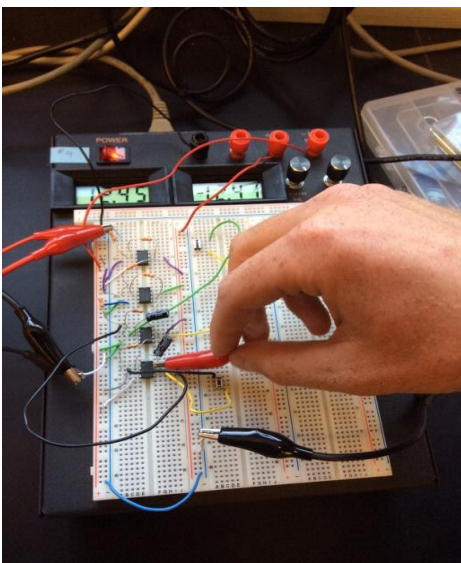

Figure 6 (The actual circuit on a bread board)

Why the Switches?

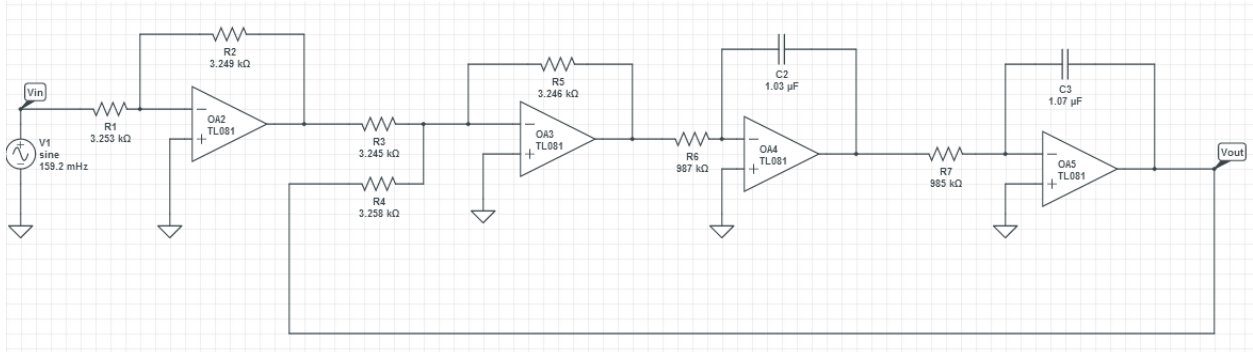Switches were used to reset the circuit in case the output hit the rails of our op amps.

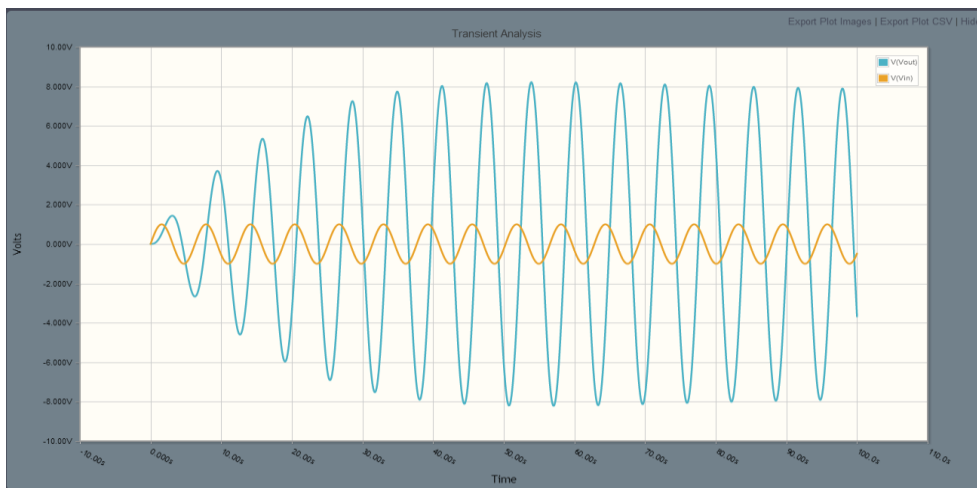**Figure 7 (Circuit Lab Schematic with Real Value Components)**

Figure 8 (Circuit Lab Plot with Real Value Components)

As you can see even with slightly different valued resistors and capacitors the output changes dramatically and is less predictable.

Once the adjusted values were taken into account and we had a graph to verify the solution. We powered up the circuit, observed the V_{in} and V_{out} voltages with an oscilloscope (shown in figure 9 -10) and compared the actual results against the theoretical outcome.
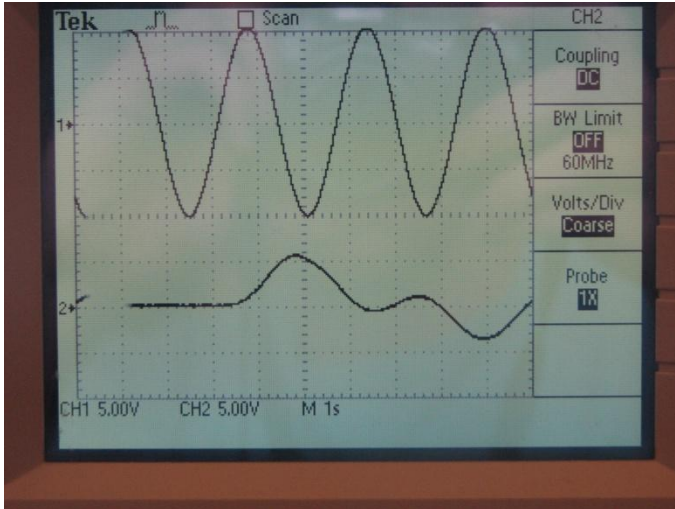
Figure 9 (Output Solution on Oscilloscope)

The top curve in figures 9 and 10 is the input sin wave and the bottom curve should be the solution to the D.E.
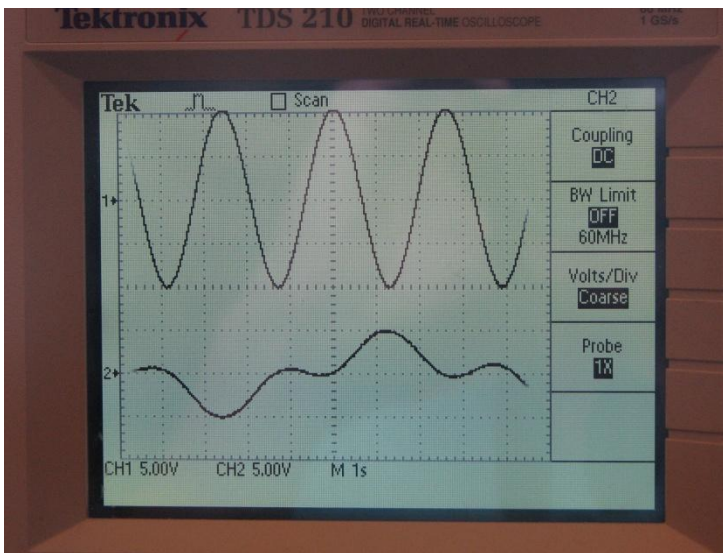

Figure 10 (Output Solution on Oscilloscope)

Notice our actual result looks nothing like our theoretical prediction. DEBUG! DEBUG! DEBUG!

"The important thing about a problem is not its solution, but rather the strength we gain in finding the solution."

Andrew and I felt this quote from our circuits textbook best summed up our experience with this project. After years of math, physics, and chemistry we were finally able to

apply what we have learned to something real…something tangible. From this project we were able to receive our first glimpse of what it means to be an engineer. Working as a team we delegated different parts of the project to each other, communicated on a regular basis, and created an actual analog computer. While the computer itself did not actually match our theoretical prediction, like the quote implies, the experience gained working as a group was invaluable to our growth as engineers. Throughout this project we learned to become better communicators, planners, designers, and circuit builders. In short, even though our project did not produce the desired results, we are stronger nonetheless.