CARTESIAN VECTORS & ADDITION & SUBTRACTION OF CARTESIAN VECTORS

Today's Objectives:

Students will be able to :

a) Represent a 3-D vector in a Cartesian coordinate system.

b) Find the magnitude and coordinate angles of a 3-D vector

c) Add vectors (forces) in 3-D space

In-Class Activities:

- Reading Quiz
- Applications / Relevance
- A Unit Vector
- 3-D Vector Terms
- Adding Vectors
- Concept Quiz
- Examples
- Attention Quiz

READING QUIZ

1. Vector algebra, as we are going to use it, is based on a ____________ coordinate system.

A) Euclidean　B) left-handed

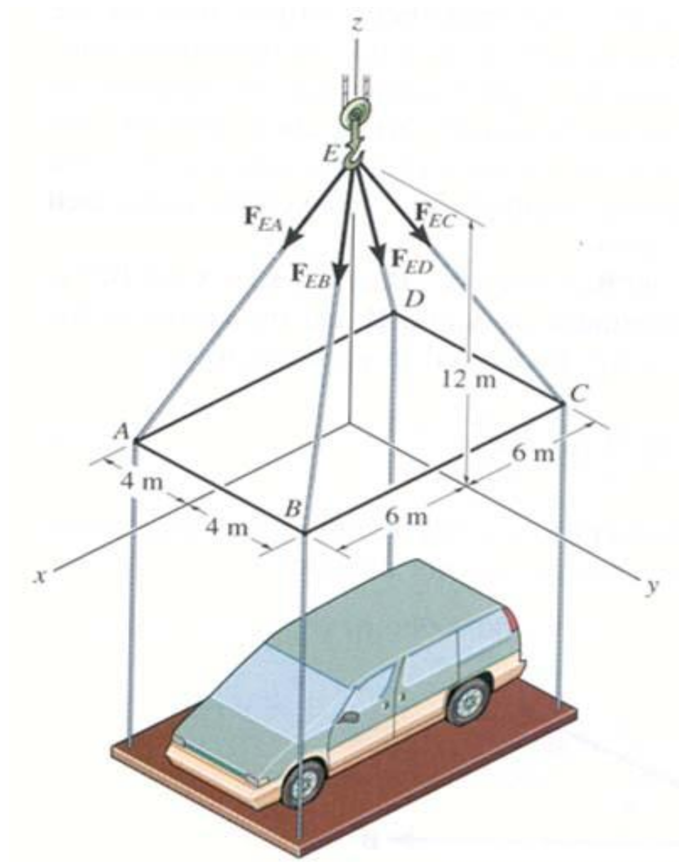
C) Greek　D) right-handed　E) Egyptian

2. The symbols α, β, and γ designate the __________ of a 3-D Cartesian vector.

A) unit vectors　B) coordinate direction angles

C) Greek societies　D) x, y and z components

APPLICATIONS

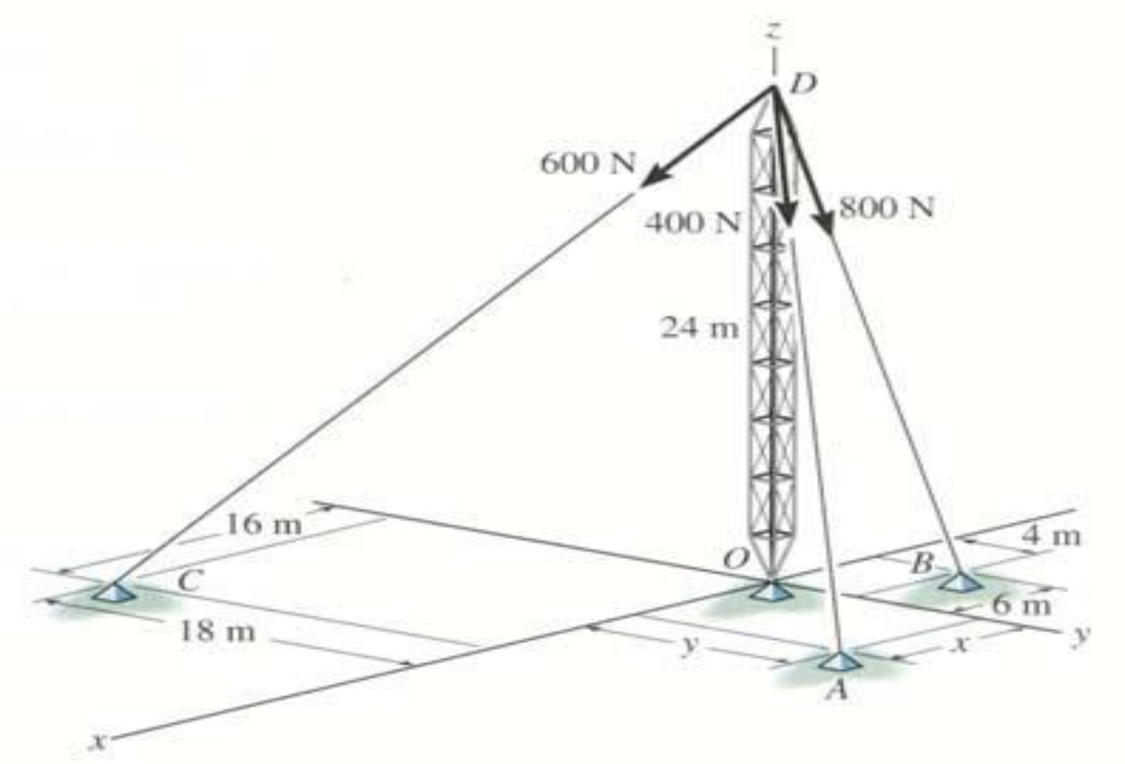


Many problems in real-life involve 3-Dimensional Space.

How will you represent each of the cable forces in Cartesian vector form?

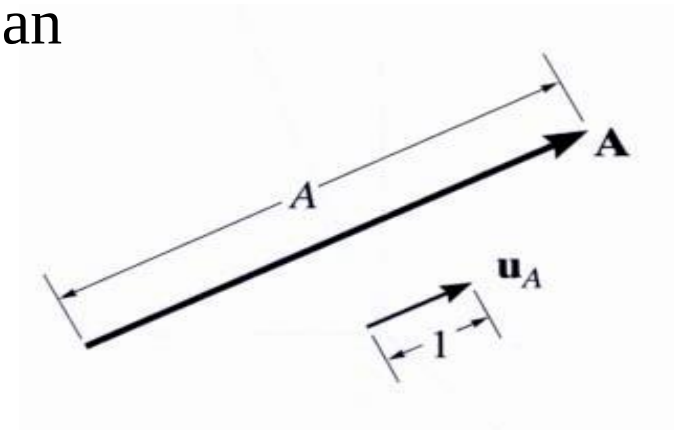
APPLICATIONS
(continued)

Given the forces in the cables, how will you determine the resultant force acting at D, the top of the tower?



A UNIT VECTOR

For a vector ***A*** with a magnitude of A, an unit vector is defined as $\boldsymbol{U}_A = \boldsymbol{A} / A$.

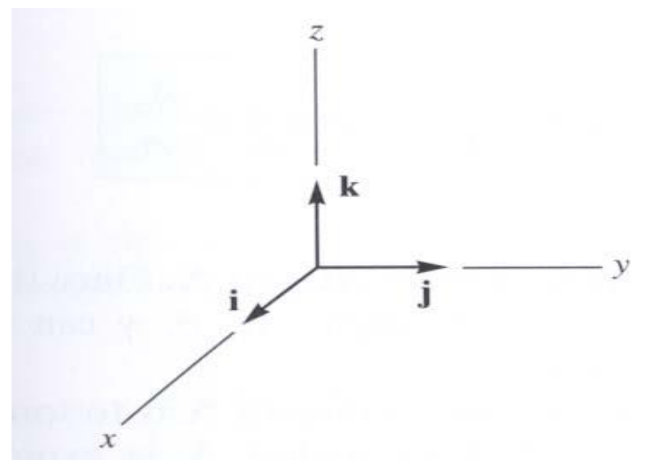


Characteristics of a unit vector**:**

a) Its magnitude is 1.

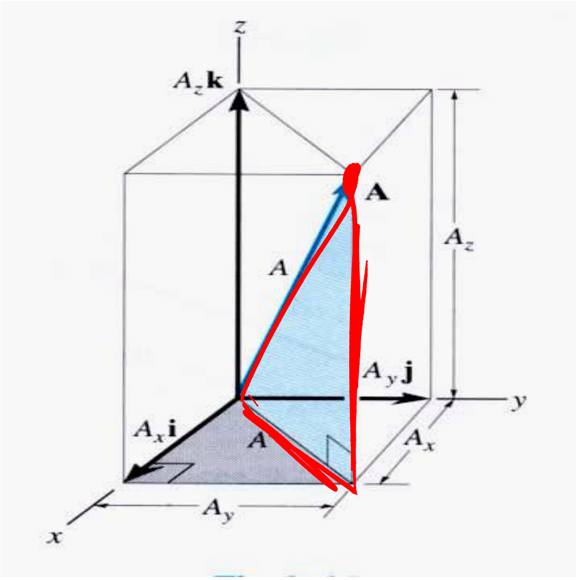
b) It is dimensionless.

c) It points in the same direction as the original vector (***A***).



The unit vectors in the Cartesian axis system are ***i***, ***j***, and ***k***. They are unit vectors along the positive x, y, and z axes respectively.

3-D CARTESIAN VECTOR TERMINOLOGY



Consider a box with sides A_X, A_Y, and A_Z meters long.

The vector ***A*** can be defined as

A = (A_X ***i*** + A_Y ***j*** + A_Z ***k***) m

The projection of the vector ***A*** in the x-y plane is A´. The magnitude of this projection, A´, is found by using the same approach as a 2-D vector: $A' = (A_X^2 + A_Y^2)^{1/2}$.

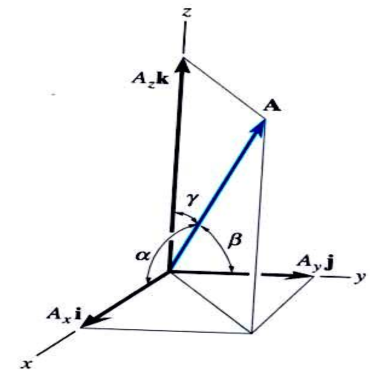
The magnitude of the position vector ***A*** can now be obtained as

$$A = ((A')^2 + A_Z^2)^{½} = (A_X^2 + A_Y^2 + A_Z^2)^{½}$$

3-D CARTESIAN VECTOR TERMINOLOGY
(continued)

The direction or orientation of vector ***A*** is defined by the angles α, β, and γ.

These angles are measured between the vector and the positive X, Y and Z axes, respectively. Their range of values are from 0° to 180°



Using trigonometry, "direction cosines" are found using the formulas

$A_x = A\cos\alpha$

$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A}$$

These angles are not independent. They must satisfy the following equation.

$$\mathbf{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

This result can be derived from the definition of a coordinate direction angles and the unit vector. Recall, the formula for finding the unit vector of any position vector:

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k}$$

or written another way, $u_A = \cos\alpha\, \boldsymbol{i} + \cos\beta\, \boldsymbol{j} + \cos\gamma\, \boldsymbol{k}$.

ADDITION/SUBTRACTION OF VECTORS
(Section 2.6)

Once individual vectors are written in Cartesian form, it is easy to add or subtract them. The process is essentially the same as when 2-D vectors are added.

For example, if

$$\boldsymbol{A} = A_X\,\boldsymbol{i} + A_Y\,\boldsymbol{j} + A_Z\,\boldsymbol{k} \quad \text{and}$$

$$\boldsymbol{B} = B_X\,\boldsymbol{i} + B_Y\,\boldsymbol{j} + B_Z\,\boldsymbol{k}, \quad \text{then}$$

$$\boldsymbol{A} + \boldsymbol{B} = (A_X + B_X)\,\boldsymbol{i} + (A_Y + B_Y)\,\boldsymbol{j} + (A_Z + B_Z)\,\boldsymbol{k}$$

or

$$\boldsymbol{A} - \boldsymbol{B} = (A_X - B_X)\,\boldsymbol{i} + (A_Y - B_Y)\,\boldsymbol{j} + (A_Z - B_Z)\,\boldsymbol{k}\,.$$

IMPORTANT NOTES

Sometimes 3-D vector information is given as:

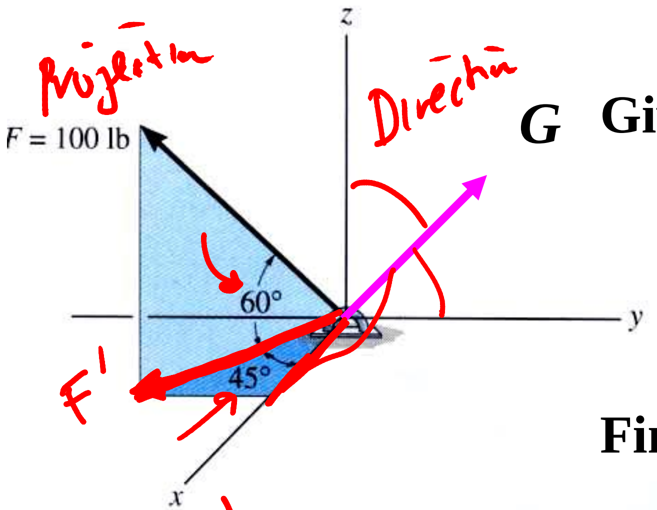
a) Magnitude and the coordinate direction angles, or

b) Magnitude and projection angles.

You should be able to use both these types of information to change the representation of the vector into the Cartesian form, i.e.,

$$\boldsymbol{F} = \{10\,\boldsymbol{i} - 20\,\boldsymbol{j} + 30\,\boldsymbol{k}\}\ \mathrm{N}\ .$$

EXAMPLE



G **Given:** Two forces ***F*** and ***G*** are applied to a hook. Force ***F*** is shown in the figure and it makes 60° angle with the X-Y plane. Force ***G*** is pointing up and has a magnitude of 80 lb with α = 111° and β = 69.3°.

Find: The resultant force in the Cartesian vector form.

$\vec{F}$ is a Projection Angle Description

$\vec{F}'$ is projection of F on x-y plane

$F' = F\cos 60^\circ = 100\cos 60^\circ = 50$

$F_x = F'\cos 45^\circ = 50\cos 45 = 35.35$

$F_y = -F'\sin 45^\circ = -50\sin 45 = -35.35$

$F_z = F\sin 60^\circ = 86.60$

$\vec{G}$ is a Direction Angle Description

$\alpha = 111^\circ, \ \beta = 69.3^\circ$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\cos\gamma = \sqrt{1 - \cos^2 111^\circ - \cos^2 69.3^\circ}$

$= .864$

$\gamma = 30.2^\circ$

$G_x = 80\cos 111^\circ = -28.66$

$G_y = 80\cos 69.3^\circ = 28.27$

$G_z = 80\cos 30.2^\circ = 69.10$

$\vec{R} = \vec{F} + \vec{G}$

$= (35.35 - 28.66)\hat{i} + (-35.35 + 28.27)\hat{j} + (86.6 + 69.1)\hat{k}$

$= 6.69\hat{i} - 7.23\hat{j} + 155.7\hat{k}$

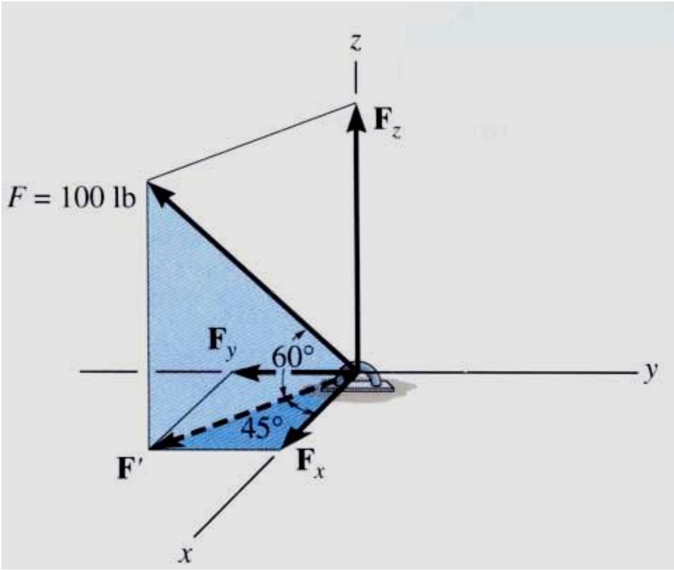
Solution : First, resolve force ***F***.

$F_z = 100 \sin 60° = 86.60$ lb

$F' = 100 \cos 60° = 50.00$ lb

$F_x = 50 \cos 45° = 35.36$ lb

$F_y = 50 \sin 45° = 35.36$ lb



Now, you can write:

$\boldsymbol{F} = \{35.36\,\boldsymbol{i} - 35.36\,\boldsymbol{j} + 86.60\,\boldsymbol{k}\}$ lb

Now resolve force ***G***.

We are given only α and β. Hence, first we need to find the value of γ.

Recall the formula $\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$.

Now substitute what we know. We have

$\cos^2(111°) + \cos^2(69.3°) + \cos^2(\gamma) = 1$.

Solving, we get γ = 30.22° or 120.2°. Since the vector is pointing up, γ = 30.22°

Now using the coordinate direction angles, we can get $\boldsymbol{U_G}$, and determine $\boldsymbol{G} = 80\ \boldsymbol{U_G}$ lb.

$\boldsymbol{G} = \{80\ (\overbrace{\cos(111°)\ \boldsymbol{i} + \cos(69.3°)\ \boldsymbol{j} + \cos(30.22°)\ \boldsymbol{k}}^{U_G})\}$ lb

$\boldsymbol{G} = \{-28.67\ \boldsymbol{i} + 28.28\ \boldsymbol{j} + 69.13\ \boldsymbol{k}\}$ lb

Now, $\boldsymbol{R} = \boldsymbol{F} + \boldsymbol{G}$ **or**

$\boldsymbol{R} = \{6.69\ \boldsymbol{i} - 7.08\ \boldsymbol{j} + 156\ \boldsymbol{k}\}$ lb

CONCEPT QUESTIONS

1. If you know just $\boldsymbol{U_A}$, you can determine the ________ of $\boldsymbol{A}$ uniquely.

   A) magnitude

   B) angles (α, β and γ)

   C) components (A_X, A_Y, & A_Z)

   D) All of the above.

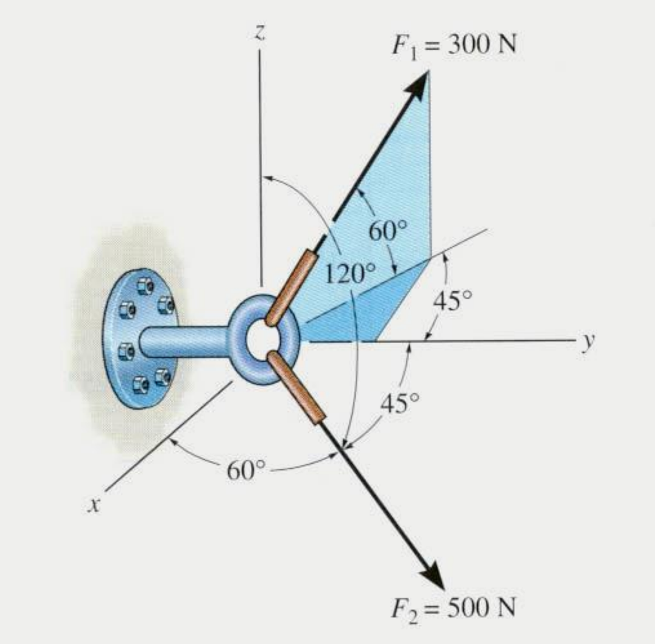
2. For an arbitrary force vector, the following parameters are randomly generated. Magnitude is 0.9 N, $\alpha = 30^{\circ}$, $\beta = 70^{\circ}$, $\gamma = 100^{\circ}$. What is <u>wrong</u> with this 3-D vector ?

   A) Magnitude is too small.

   B) Angles are too large.

   C) All three angles are arbitrarily picked.

   D) All three angles are between 0° to 180°.

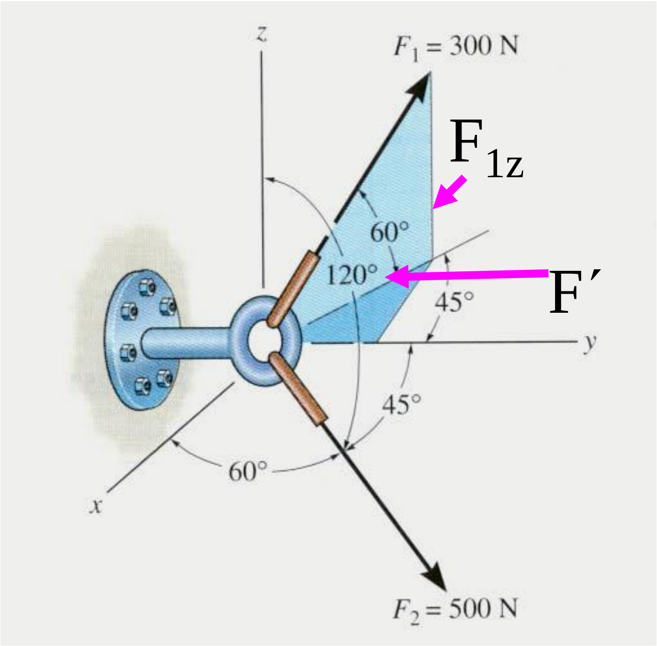


GROUP PROBLEM SOLVING

Given: The screw eye is subjected to two forces.

Find: The magnitude and the coordinate direction angles of the resultant force.

GROUP PROBLEM SOLVING (continued)



First resolve the force $\boldsymbol{F_1}$.

F_{1z} = 300 sin 60° = 259.8 N

F´ = 300 cos 60° = 150.0 N

F' can be further resolved as,

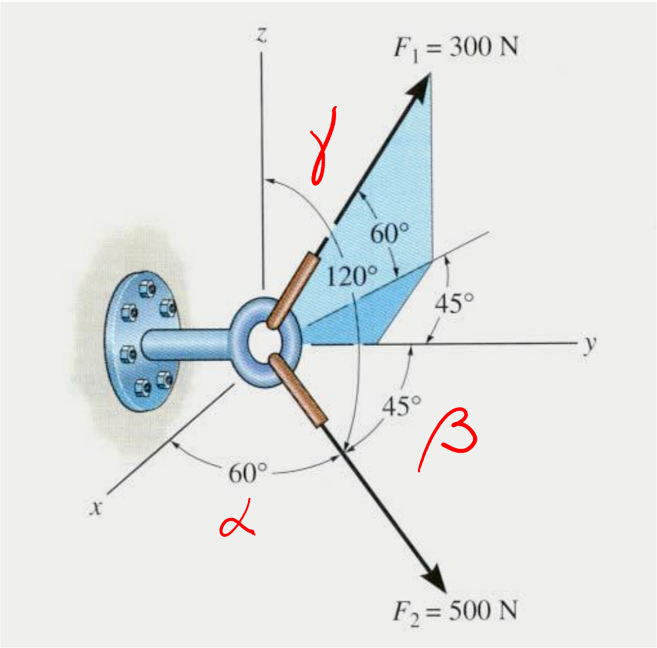
F_{1x} = -150 sin 45° = -106.1 N

F_{1y} = 150 cos 45° = 106.1 N

Now we can write :

$\boldsymbol{F_1}$ = {-106.1 $\boldsymbol{i}$ + 106.1 $\boldsymbol{j}$ + 259.8 $\boldsymbol{k}$ } N

GROUP PROBLEM SOLVING (continued)



The force $\boldsymbol{F_2}$ can be represented in the Cartesian vector form as:

$$\boldsymbol{F_2} = 500\{\cos 60°\,\boldsymbol{i} + \cos 45°\,\boldsymbol{j} + \cos 120°\,\boldsymbol{k}\}\text{ N}$$

$$= \{250\,\boldsymbol{i} + 353.6\,\boldsymbol{j} - 250\,\boldsymbol{k}\}\text{ N}$$

$$\boldsymbol{F_R} = \boldsymbol{F_1} + \boldsymbol{F_2}$$

$$= \{143.9\,\boldsymbol{i} + 459.6\,\boldsymbol{j} + 9.81\,\boldsymbol{k}\}\text{ N}$$

$$F_R = (143.9^2 + 459.6^2 + 9.81^2)^{½} = 481.7 = 482\text{ N}$$

$$\alpha = \cos^{-1}(F_{Rx} / F_R) = \cos^{-1}(143.9/481.7) = 72.6°$$

$$\beta = \cos^{-1}(F_{Ry} / F_R) = \cos^{-1}(459.6/481.7) = 17.4°$$

$$\gamma = \cos^{-1}(F_{Rz} / F_R) = \cos^{-1}(9.81/481.7) = 88.8°$$

ATTENTION QUIZ

1. What is not true about an unit vector, $\boldsymbol{U_A}$ **?**

   A) It is dimensionless.

   B) Its magnitude is one.

   C) It always points in the direction of positive X- axis.

   D) It always points in the direction of vector ***A***.

2. If ***F*** = {10 ***i*** + 10 ***j*** + 10 ***k***} N and ***G*** = {20 ***i*** + 20 ***j*** + 20 ***k*** } N, then ***F*** + ***G*** = { _____ } **N**

   A) 10 ***i*** + 10 ***j*** + 10 ***k***

   B) 30 ***i*** + 20 ***j*** + 30 ***k***

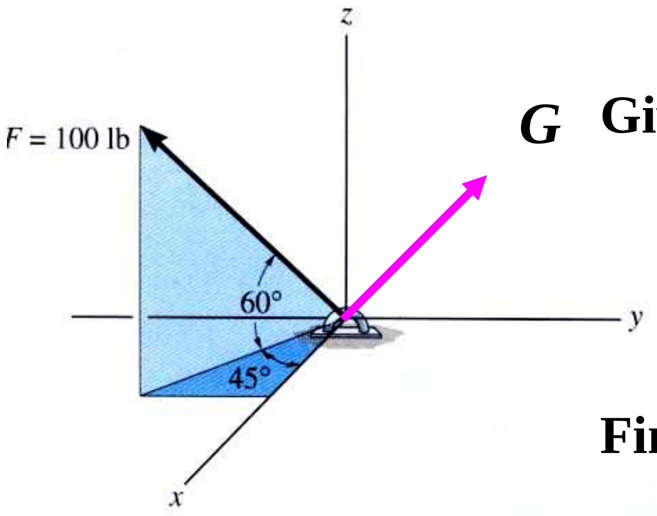
   C) -10 ***i*** - 10 ***j*** - 10 ***k***

   D) 30 ***i*** + 30 ***j*** + 30 ***k***

End of the Lecture

Let Learning Continue

EXAMPLE



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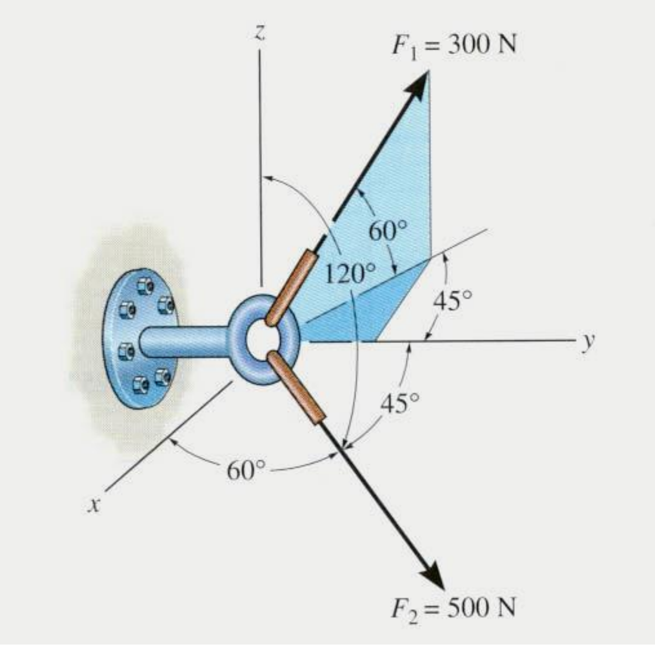

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