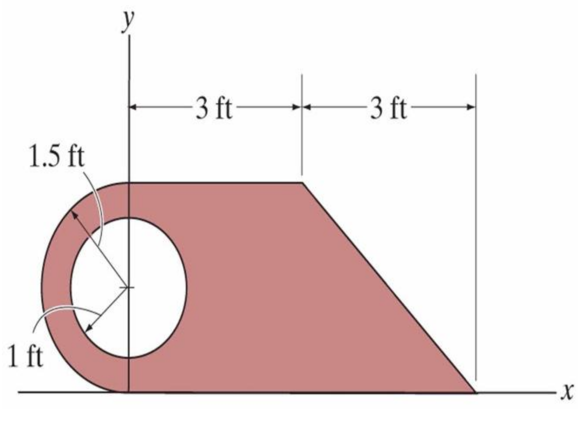
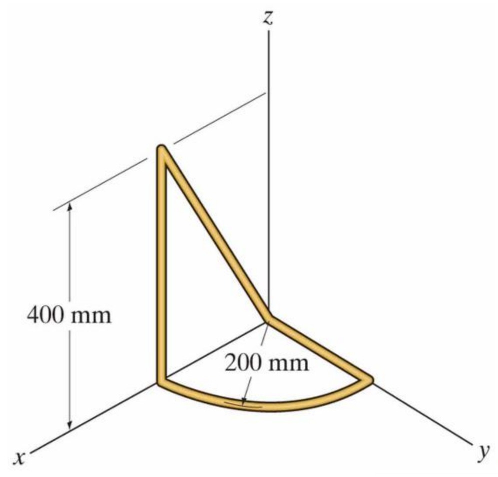
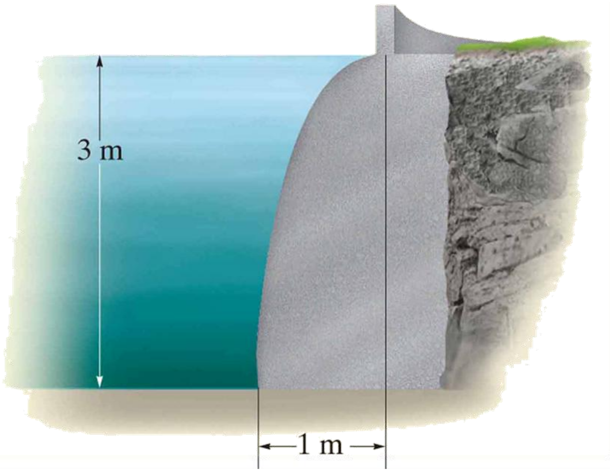
Centroidal Analysis (1st Moments)

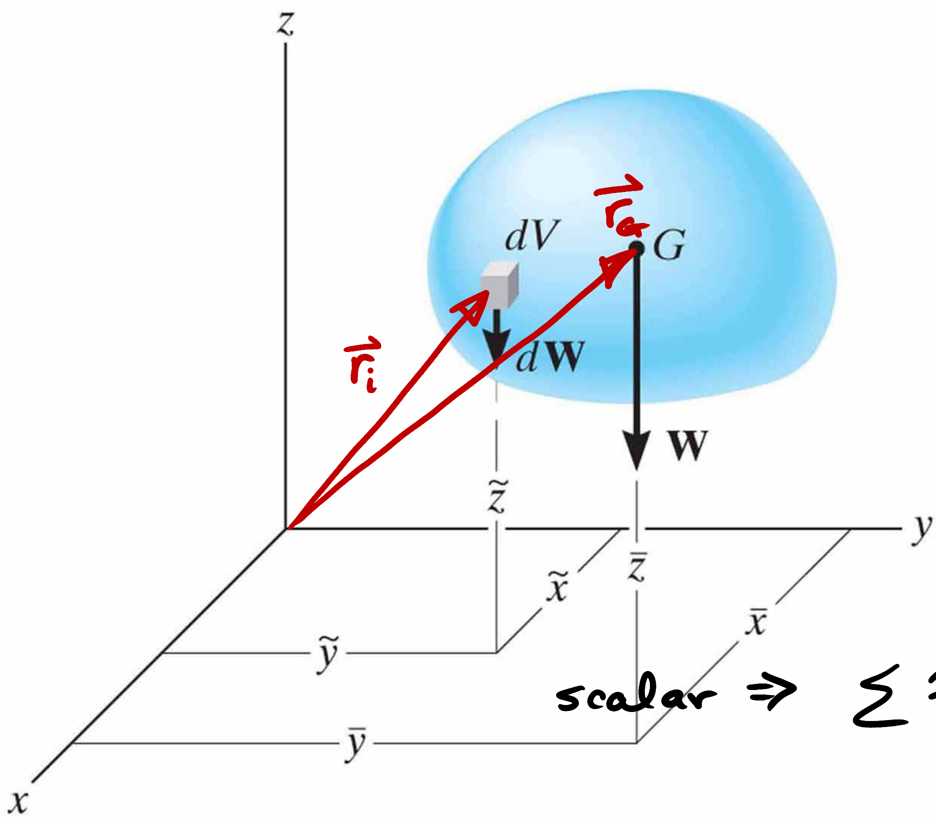






- **Center of Gravity, Center of Mass, & Centroid**
- **Centroids of V, A, & L by Integration**
- **Composite Body Analysis**
- **Theorems of Pappus-Guldinus**
- **Hydrostatic Pressure Analysis**

Center of Gravity



Equivalent Force Systems

$$\sum_1 \vec{F} = \sum_2 \vec{F} \Rightarrow \sum dW_i = W$$

$$\sum_1 \vec{M}_O = \sum_2 \vec{M}_O$$

$$\sum \vec{r}_i \times d\vec{W}_i = \vec{r}_G \times \vec{W}$$

scalar $\Rightarrow \sum \tilde{x}_i dW_i = \bar{x} W$; $\sum \tilde{y}_i dW_i = \bar{y} W$

<u>integral form</u> $W = \int dW$

$$\bar{x} = \int \tilde{x}_i dW / \int dW$$

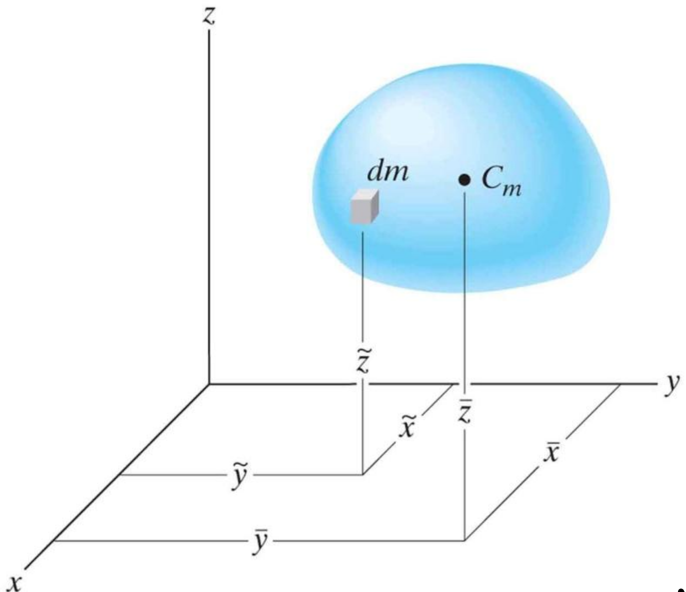
$$\bar{y} = \int \tilde{y}_i dW / \int dW$$

$$\bar{z} = \int \tilde{z}_i dW / \int dW$$

Center of Mass and Centroid

$W = mg \Rightarrow dW = g\,dm$ (g usually constant)

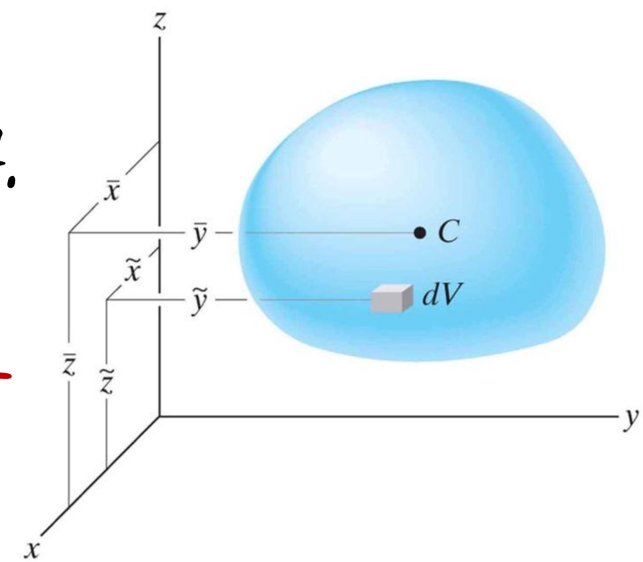
$$\bar{x} = \frac{\int \tilde{x}\,dW}{\int dW} = \frac{\int \tilde{x}\,\cancel{g}\,dm}{\int \cancel{g}\,dm}$$



$$\Rightarrow \quad \bar{x} = \int \tilde{x}\,dm \Big/ \int dm \qquad \text{C.O.M.}$$

$m = \rho V \Rightarrow dm = \rho\,dV$ if $\rho = \text{const.}$

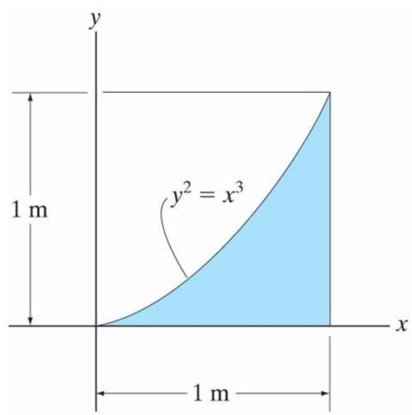
$$\Rightarrow \quad \bar{x} = \int \tilde{x}\,dV \Big/ \int dV \qquad \text{centroid}$$



2D

Centroids of Areas

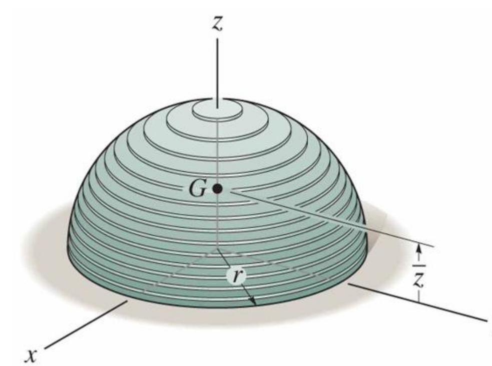
3D



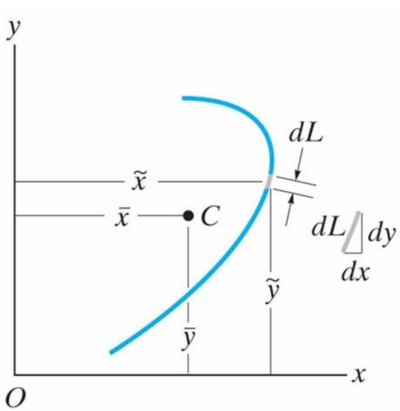
$$\bar{x} = \int \tilde{x}\, dA \Big/ \int dA$$

$$\bar{y} = \int \tilde{y}\, dA \Big/ \int dA$$

$$\bar{z} = \int \tilde{z}\, dA \Big/ \int dA$$



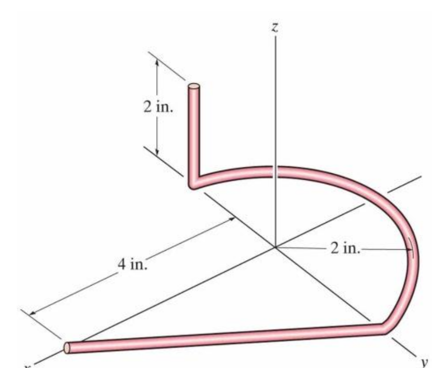
Centroids of Lines



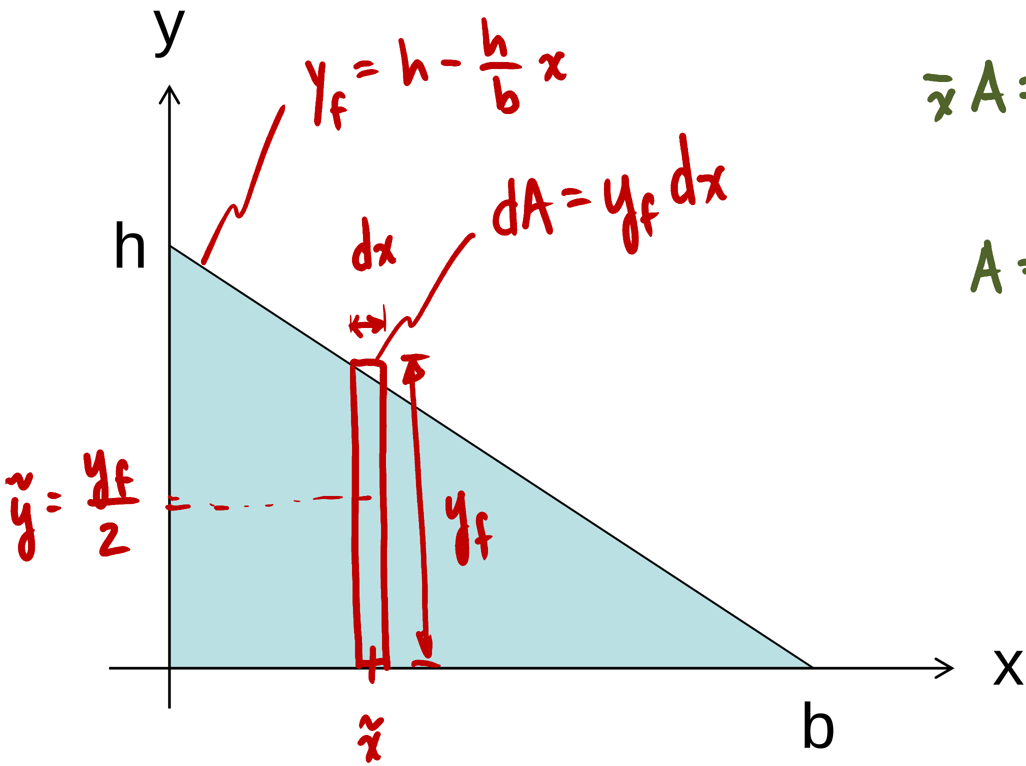
$$\bar{x} = \int \tilde{x}\, dL \Big/ \int dL$$

$$\bar{y} = \int \tilde{y}\, dL \Big/ \int dL$$

$$\bar{z} = \int \tilde{z}\, dL \Big/ \int dL$$



Centroid by Integration: Area Example



$$\bar{x}A = \int \tilde{x}\,dA$$

$$A = \int dA = \int_0^b y_f\,dx = \int_0^b h - \frac{h}{b}x\,dx$$

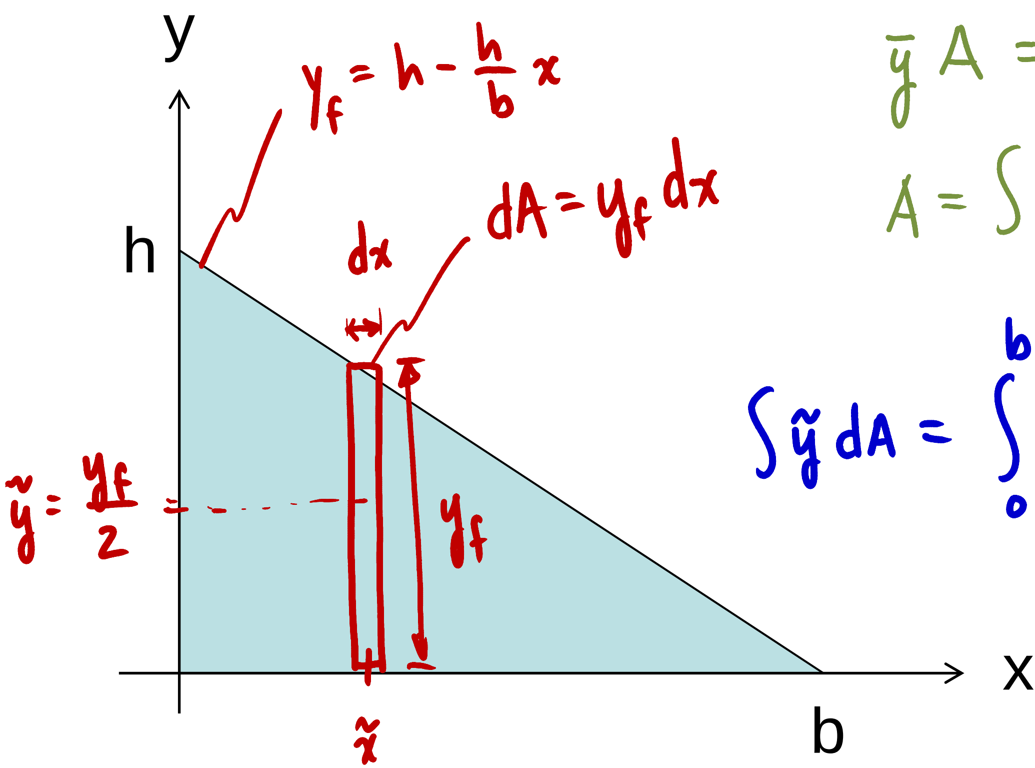
$$= \left[hx - \frac{h}{b}\frac{x^2}{2}\right]_0^b = hb - \frac{hb^{\cancel{2}}}{2\cancel{b}}$$

$$\Rightarrow A = \frac{1}{2}bh$$

$$\int \tilde{x}\,dA = \int_0^b x\,y_f\,dx = \int_0^b hx - \frac{h}{b}x^2\,dx = \left[\frac{hx^2}{2} - \frac{hx^3}{3b}\right]_0^b = \frac{hb^2}{2} - \frac{hb^{\cancel{3}\,2}}{3\cancel{b}} = \frac{1}{6}hb^2$$

$$\bar{x} = \frac{\int \tilde{x}\,dA}{\int dA} = \frac{\frac{1}{6}\cancel{h}b^{\cancel{2}}}{\frac{1}{2}\cancel{b}\cancel{h}} \Rightarrow \bar{x} = \frac{1}{3}b$$

Centroid by Integration: Area Example



$$\bar{y} A = \int \tilde{y}\, dA$$

$$A = \int dA = \frac{1}{2} bh$$

$$\int \tilde{y}\, dA = \int_0^b \frac{y_f}{2} y_f\, dx = \int_0^b \frac{1}{2}\left(h - \frac{h}{b}x\right)^2 dx$$

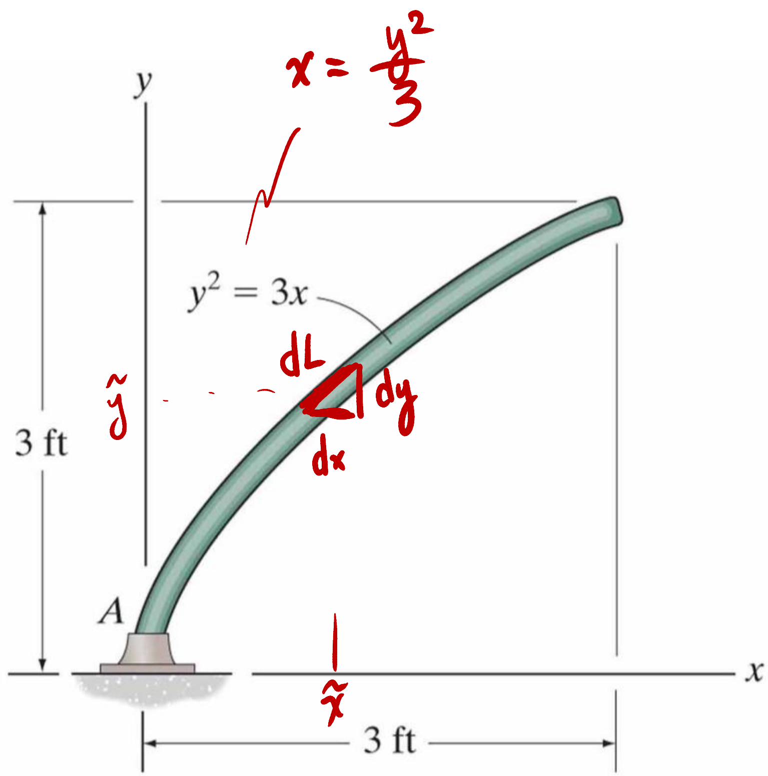
$$= \frac{1}{2}\int_0^b h^2 - 2\frac{h^2}{b}x + \frac{h^2}{b^2}x^2\, dx$$

$$= \frac{1}{2}\left[h^2 x - \frac{h^2}{b}x^2 + \frac{h^2}{3b^2}x^3\right]_0^b = \frac{1}{2}\left[h^2 b - \frac{h^2 b^{\cancel{2}}}{\cancel{b}} + \frac{h^2 b^{\cancel{3}}}{3\cancel{b^2}}\right] = \frac{1}{6}h^2 b$$

$$\bar{y} = \frac{\int \tilde{y}\, dA}{A} = \frac{\frac{1}{6}h^{\cancel{2}}\cancel{b}}{\frac{1}{2}\cancel{b}\cancel{h}} \Rightarrow \qquad \bar{y} = \frac{1}{3}h$$

Centroid by Integration: Line Example

Find $\bar{y}$



$$\bar{y} = \int \tilde{y}\, dL \Big/ \int dL$$

$$dL = \sqrt{dx^2 + dy^2} = \sqrt{\left(dx^2/dy^2 + 1\right) dy^2}$$

$$= \sqrt{\left(dx/dy\right)^2 + 1}\; dy$$

$$\frac{dx}{dy} = \frac{2y}{3} \Rightarrow dL = \sqrt{\frac{4y^2}{9} + 1}\; dy$$

$$L = \int dL = \int_0^3 \sqrt{4/9\, y^2 + 1}\; dy$$

integral tables $\int \sqrt{x^2 + a^2}\, dx = \ldots$

numerical integration ...

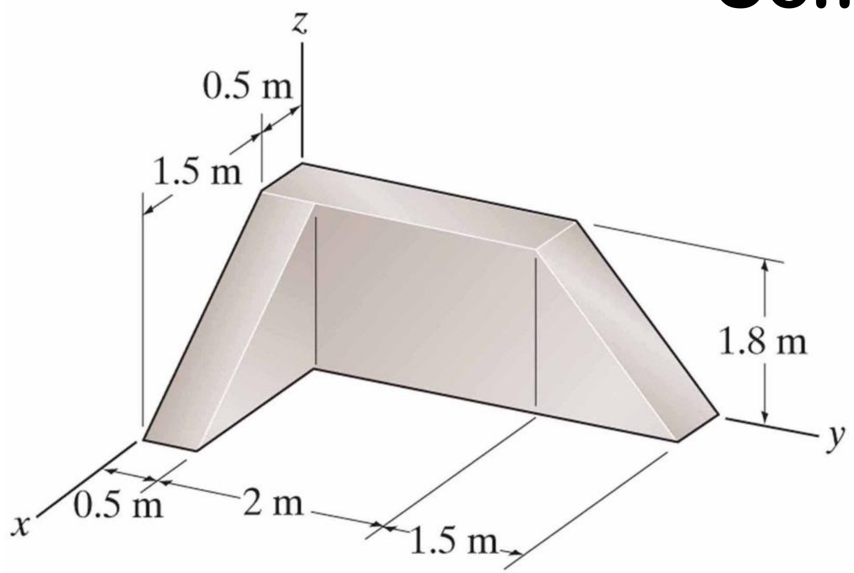
$\Rightarrow$

$$\int \tilde{y}\, dL = \int_0^3 y \sqrt{4/9\, y^2 + 1}\; dy = \frac{3}{4}\left(\frac{4}{9} y^2 + 1\right)^{3/2} \Big|_0^3 =$$

$$\bar{y} = \frac{\int \tilde{y}\, dL}{L}$$

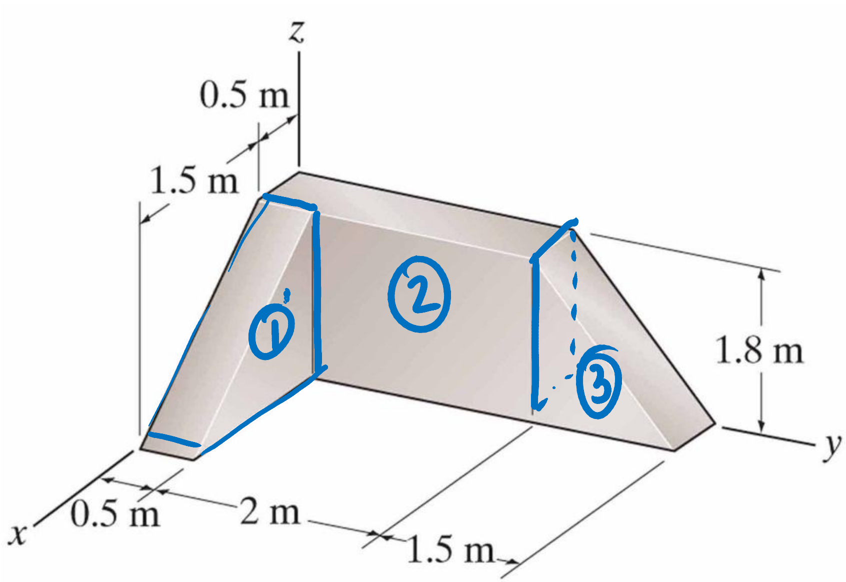
$\Rightarrow$

Composite Body Approach



$$W = \sum W_i$$

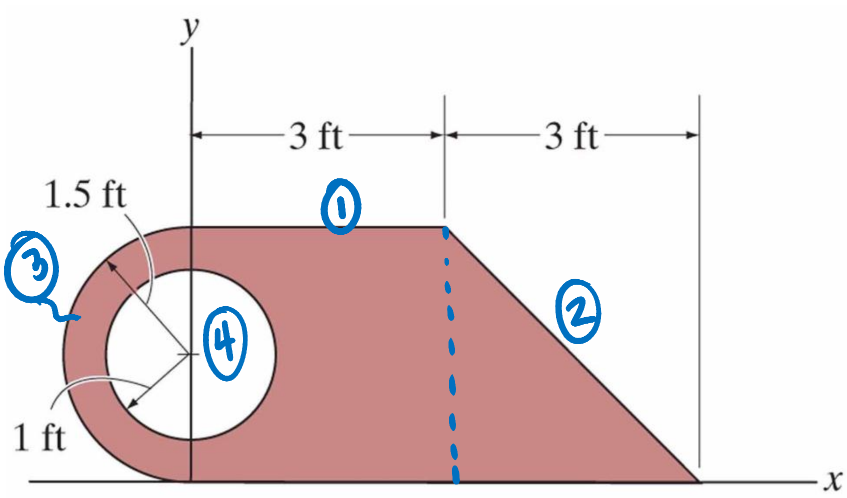
$$\bar{x} W = \sum \tilde{x}_i W_i$$

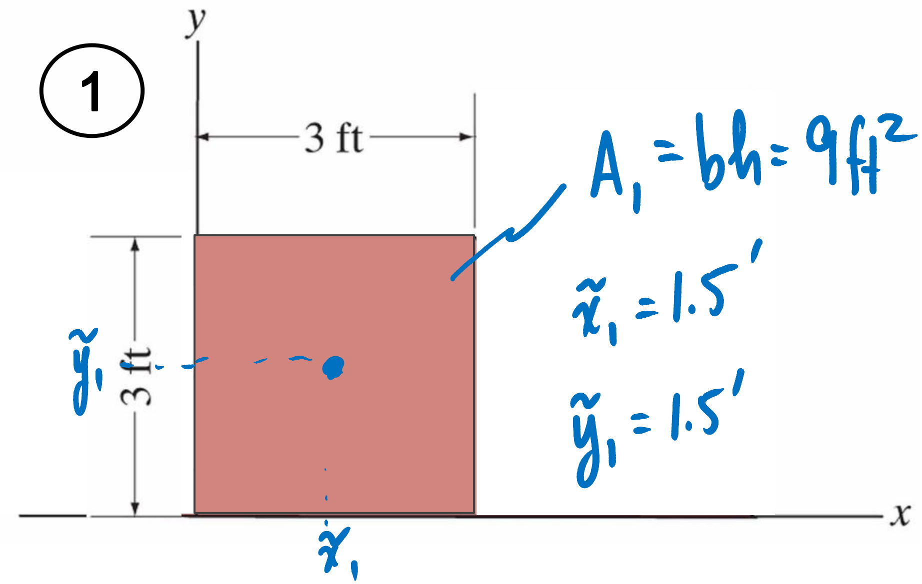


$$V = V_1 + V_2 + V_3$$

$$\bar{x} V = \tilde{x}_1 V_1 + \tilde{x}_2 V_2 + \tilde{x}_3 V_3$$

Composite Body Example





$A_1 = bh = 9\,ft^2$

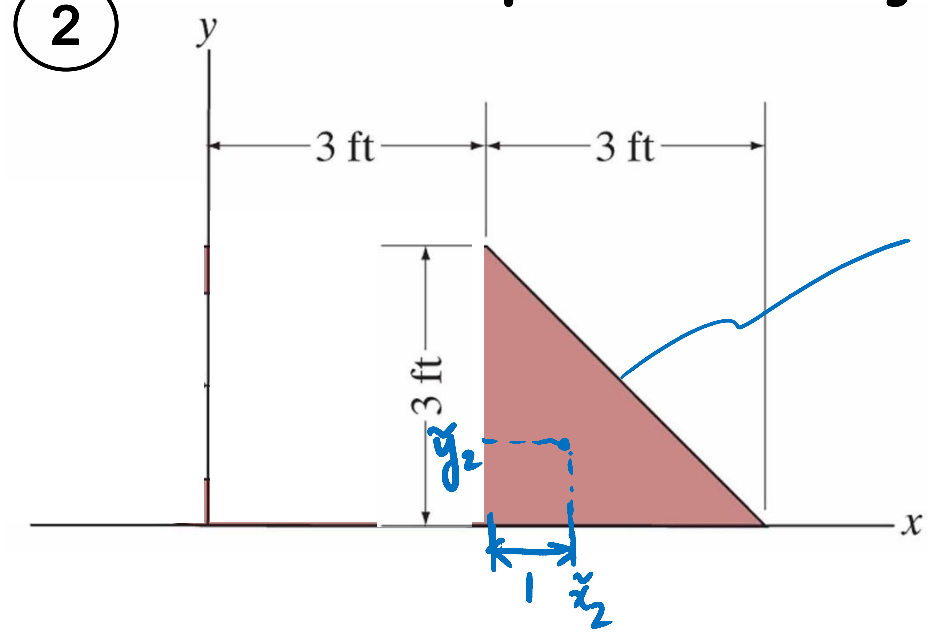
$\tilde{x}_1 = 1.5'$

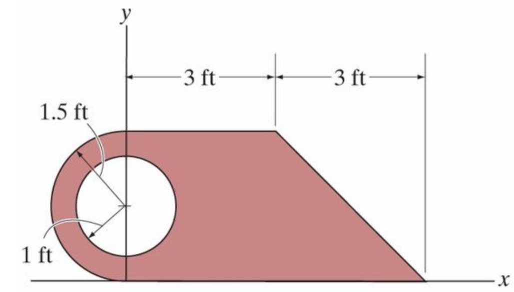
$\tilde{y}_1 = 1.5'$

#	$\tilde{x}_i$	$\tilde{y}_i$	A_i	$\tilde{x}_i A_i$	$\tilde{y}_i A_i$
1	1.5	1.5	9	13.5	13.5
2					
3					
4					

2

Composite Body Example



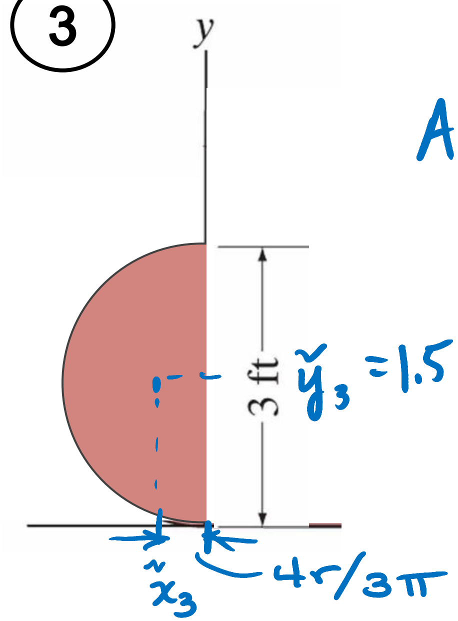


$$A_2 = \frac{1}{2}bh$$
$$= \frac{1}{2}(3)(3) = 4.5\ ft^2$$
$$\tilde{x}_2 = 3 + \frac{1}{3}3 = 4$$
$$\tilde{y}_2 = \frac{1}{3}(3) = 1$$

#	$\tilde{x}_i$	$\tilde{y}_i$	A_i	$\tilde{x}_i A_i$	$\tilde{y}_i A_i$
1	1.5	1.5	9	13.5	13.5
2	4	1	4.5	18	4.5
3					
4					

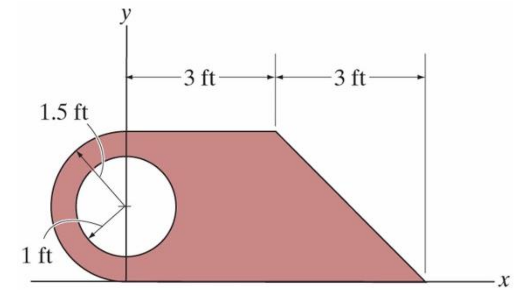
Composite Body Example

3



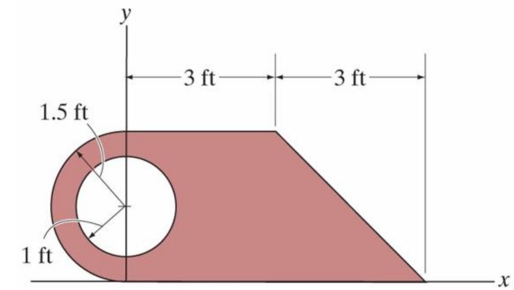
$$A_3 = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(1.5)^2 = 3.53 \text{ ft}^2$$

$$\tilde{x}_3 = 0 - \frac{4r}{3\pi} = -\frac{4(1.5)}{3\pi} = -0.637 \text{ in}$$

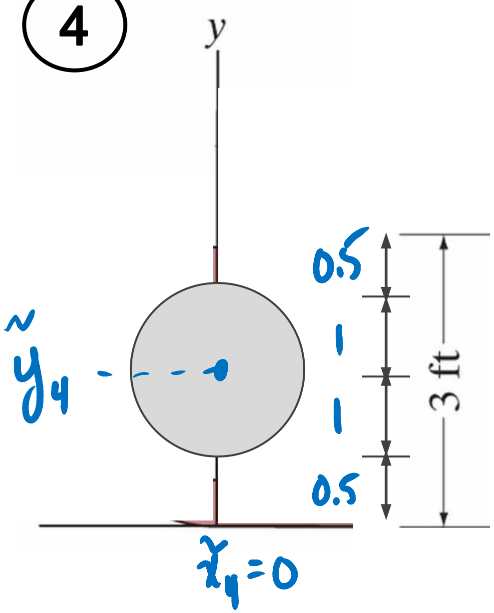


#	$\tilde{x}_i$	$\tilde{y}_i$	A_i	$\tilde{x}_i A_i$	$\tilde{y}_i A_i$
1	1.5	1.5	9	13.5	13.5
2	4	1	4.5	18	4.5
3	-0.637	1.5	3.53	-2.25	5.30
4					

Composite Body Example



4



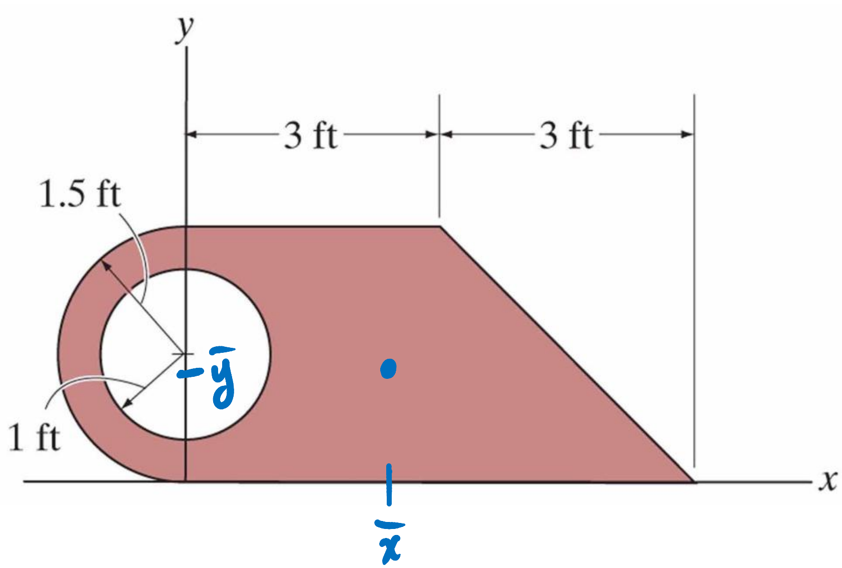
$$A_4 = -\pi r^2 = -\pi (1)^2$$

$$\tilde{x}_4 = 0$$

$$\tilde{y}_4 = 1.5$$

#	$\tilde{x}_i$	$\tilde{y}_i$	A_i	$\tilde{x}_i A_i$	$\tilde{y}_i A_i$
1	1.5	1.5	9	13.5	13.5
2	4	1	4.5	18	4.5
3	–0.637	1.5	3.53	–2.25	5.30
4	0	1.5	-3.14	0	-4.71

Composite Body Example



$$\bar{x}A = \sum \tilde{x}_i A_i \Rightarrow \bar{x} = \frac{\sum \tilde{x}_i A_i}{\sum A_i}$$

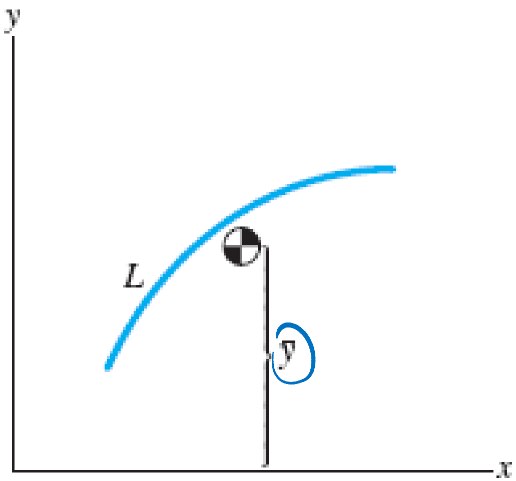
$$\bar{x} = \frac{29.25}{13.89} = \underline{\underline{2.11 \text{ in}}}$$

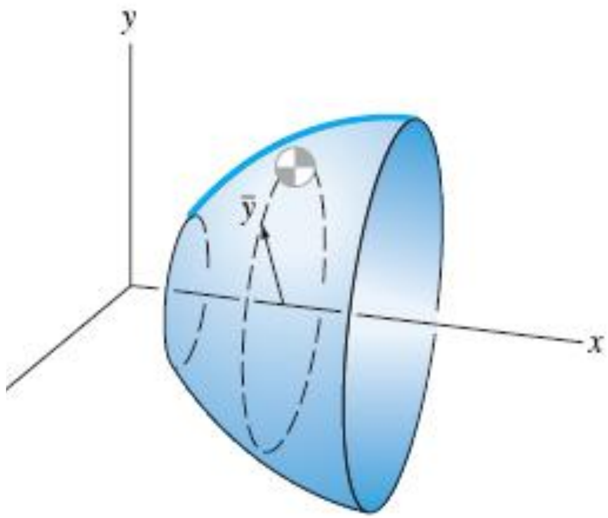
$$\bar{y} = \frac{\sum \tilde{y}_i A_i}{\sum A_i} = \frac{18.59}{13.89} = \underline{\underline{1.34 \text{ in}}}$$

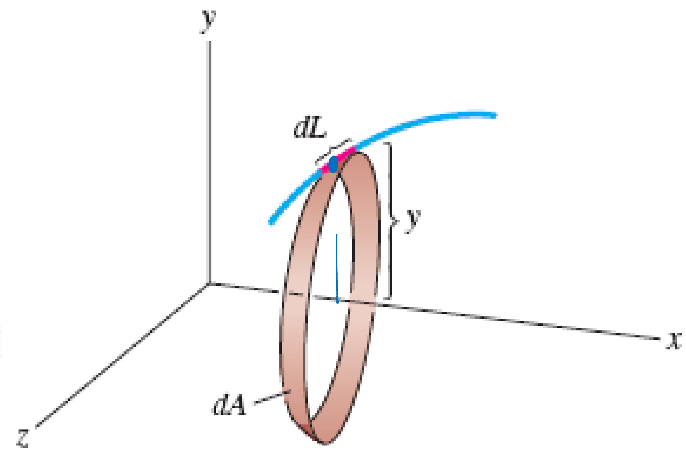
#	$\tilde{x}_i$	$\tilde{y}_i$	A_i	$\tilde{x}_i A_i$	$\tilde{y}_i A_i$
1	1.5	1.5	9	13.5	13.5
2	4	1	4.5	18	4.5
3	–0.637	1.5	3.53	- –2.25	5.30
4	0	1.5	–3.14	0	–4.71
			13.89	29.25	18.59

Pappus-Guldinus Theorems

First Theorem:







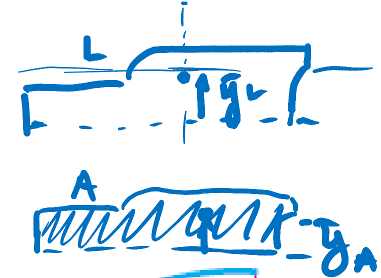
- Given: line L, with centroid
- Rotate L about the x-axis → Surface S
- Find S

$$dA = 2\pi y \, dL$$

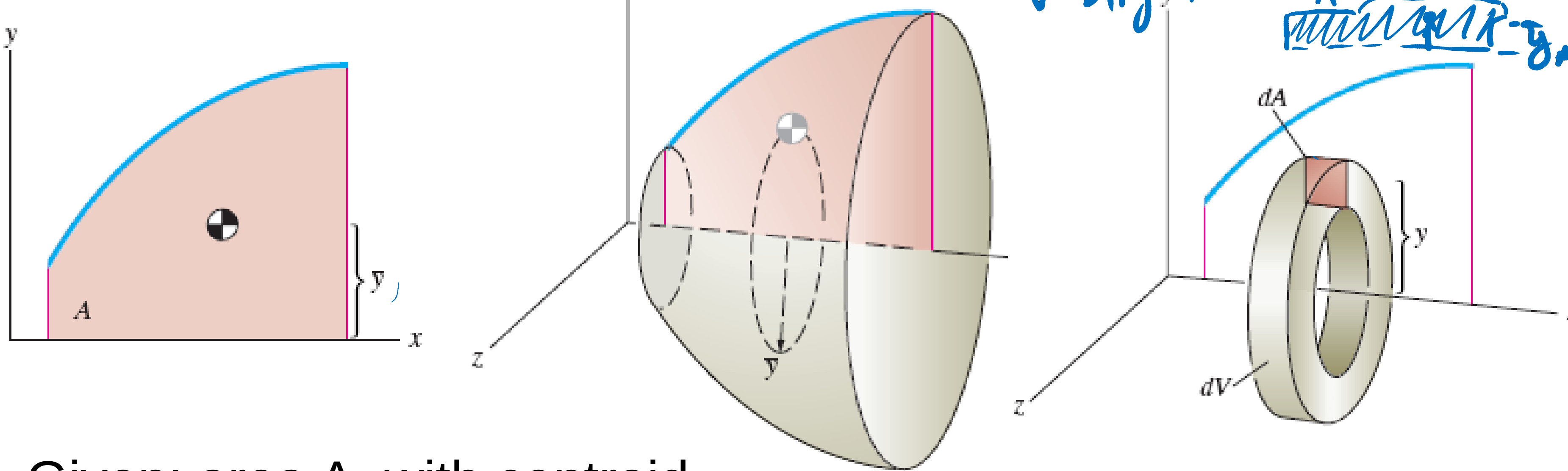
$$S = \int dA = \int 2\pi y \, dL = 2\pi \int y \, dL = 2\pi \bar{y} L$$

Second Theorem:

$S = 2\pi \bar{y} L$



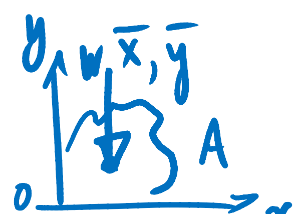
$V = 2\pi \bar{y} A$



- Given: area A, with centroid
- Rotate A about the x-axis → Volume
- Find the volume V.

$dV = 2\pi y \, dA$

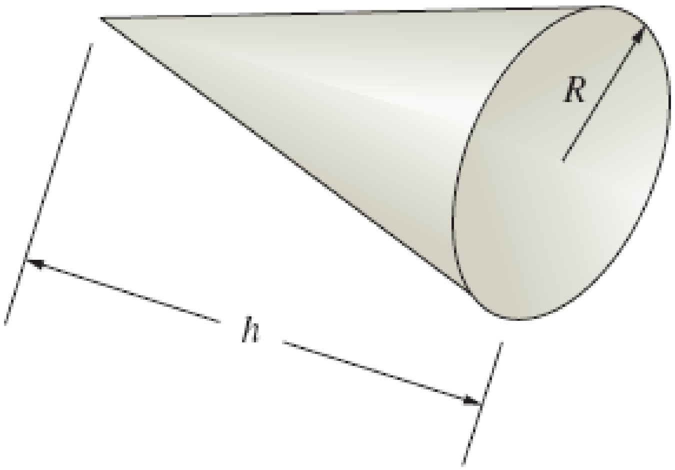
$$V = \int dV = \int 2\pi y \, dA = 2\pi \int y \, dA = \underline{\underline{2\pi \bar{y} A}}$$

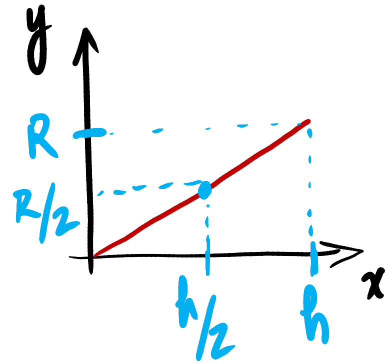


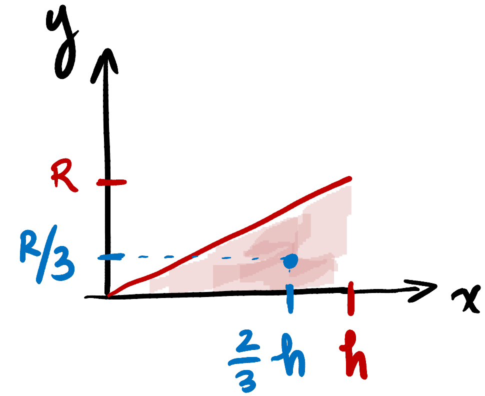
$\bar{x} A = \sum x_i A_i$

$= \int x \, dA$

Ex. 7. Use Pappus-Guldinus Theorems to find the surface area and volume of a cone.



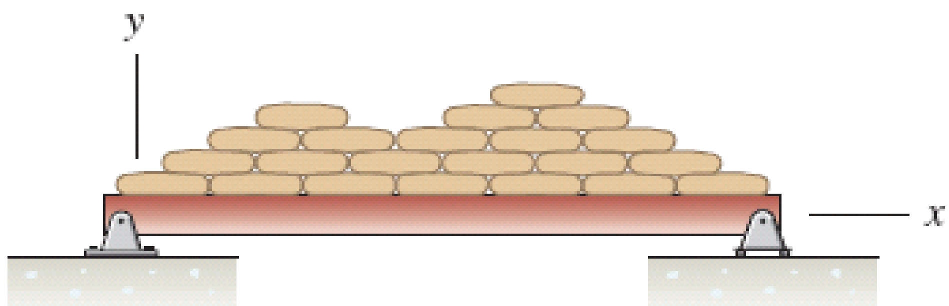


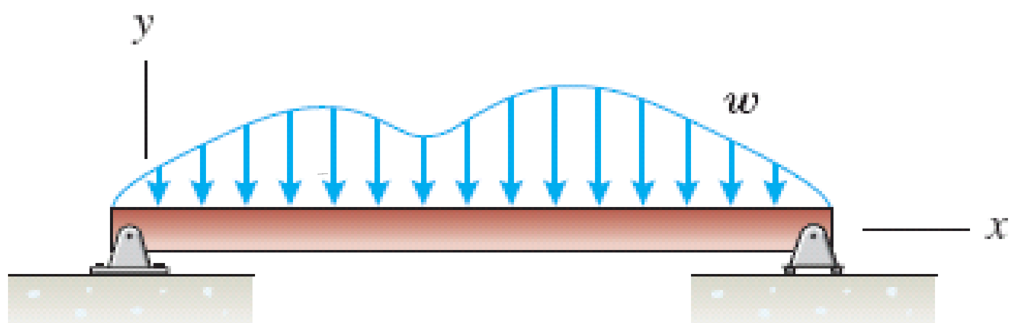


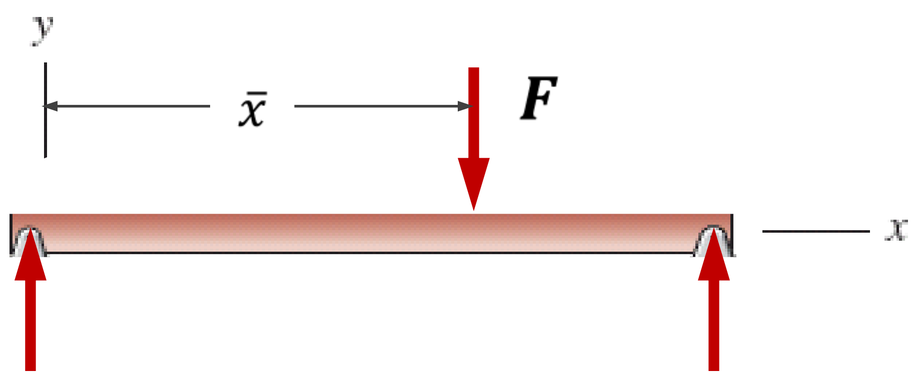
$$S = 2\pi \bar{y} L = \cancel{2}\pi\left(\frac{R}{\cancel{2}}\right)\sqrt{R^2+h^2} = \pi R\sqrt{R^2+h^2} \qquad \left(+\pi R^2\right)$$

$$V = 2\pi \bar{y} A = \cancel{2}\pi\left(\frac{R}{3}\right)\left(\frac{1}{\cancel{2}}Rh\right) = \frac{1}{3}\pi R^2 h$$

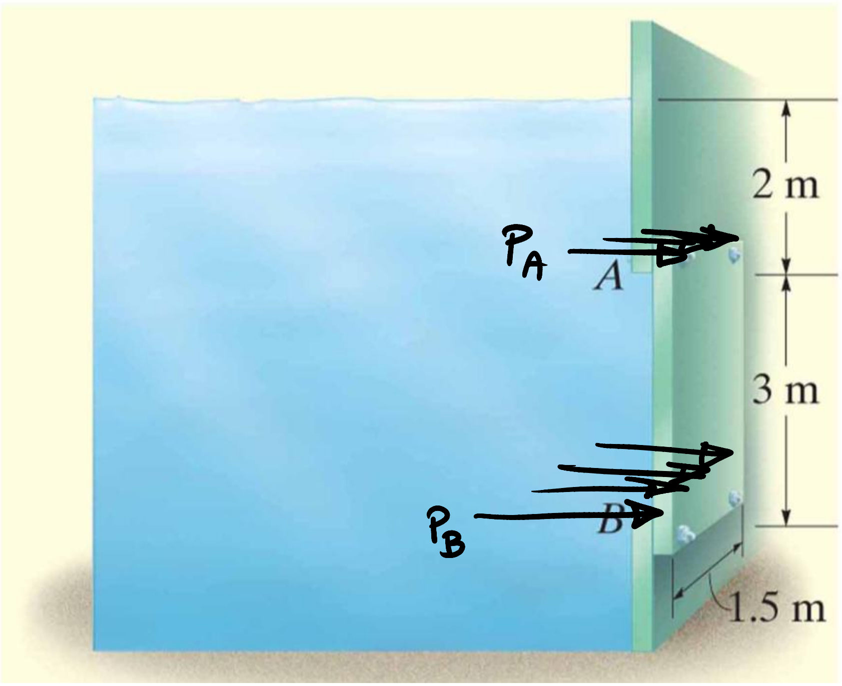
Distributed Loading

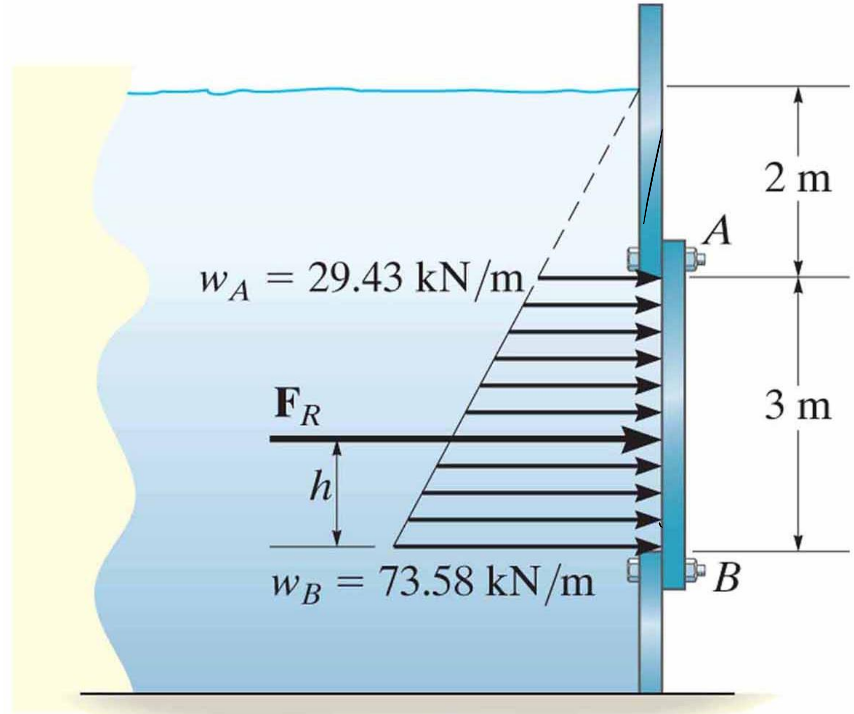






Hydrostatic Pressure Loading





Fluid pressure varies with depth:

1000 kg/m³

$$p = \rho g z$$

9.81 m/s²

Convert pressure into load intensity:

1.5m

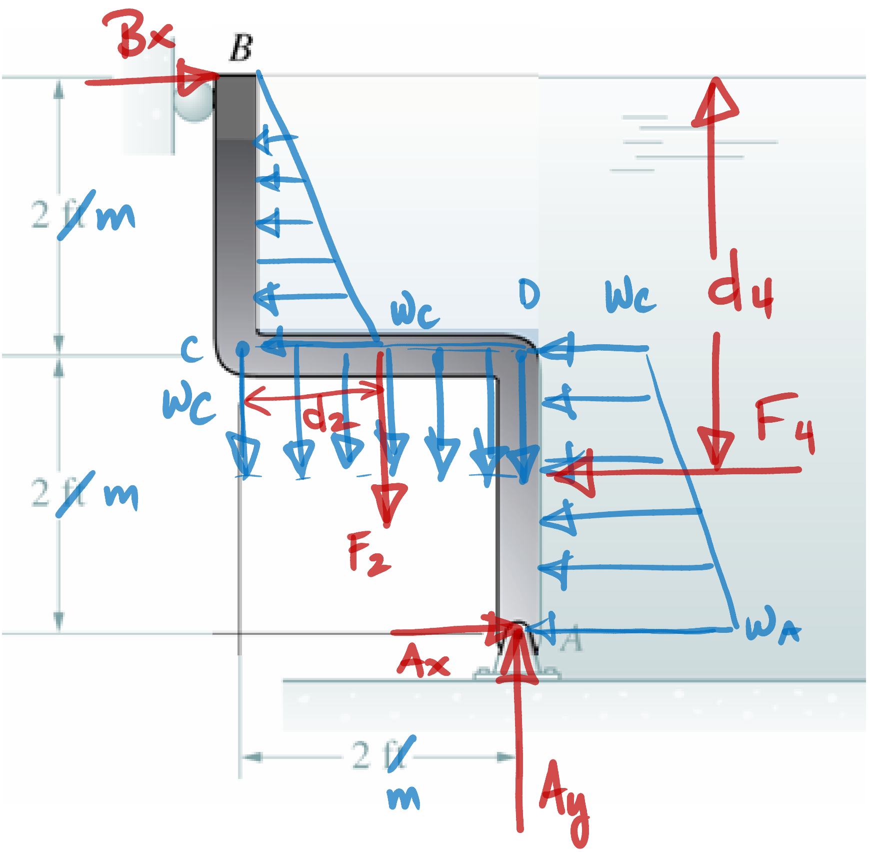
$$w = \rho g b z$$

F_R = Area of load diagram

h = centroid of " "

Hydrostatic Pressure Example

Determine the reactions at A and B for the 10 ~~ft~~ m wide gate.



$$F_4 = F_1 + F_3$$

$$d_4 F_4 = d_1 F_1 + d_3 F_3$$

$$w_c = \rho g b d_c = 1000 \frac{kg}{m^3}\left(9.81 \frac{m}{s^2}\right)(10\,m)(2\,m)$$

$$w_c = 196.2\ kN/m$$

$$w_A = 2 w_c = 392.4\ kN/m$$

$$F_4 = \frac{1}{2} bh = \frac{1}{2} w_A (4m) = 784.8\ kN$$

$$d_4 = \frac{2}{3}(4\,m) = \frac{8}{3} m$$

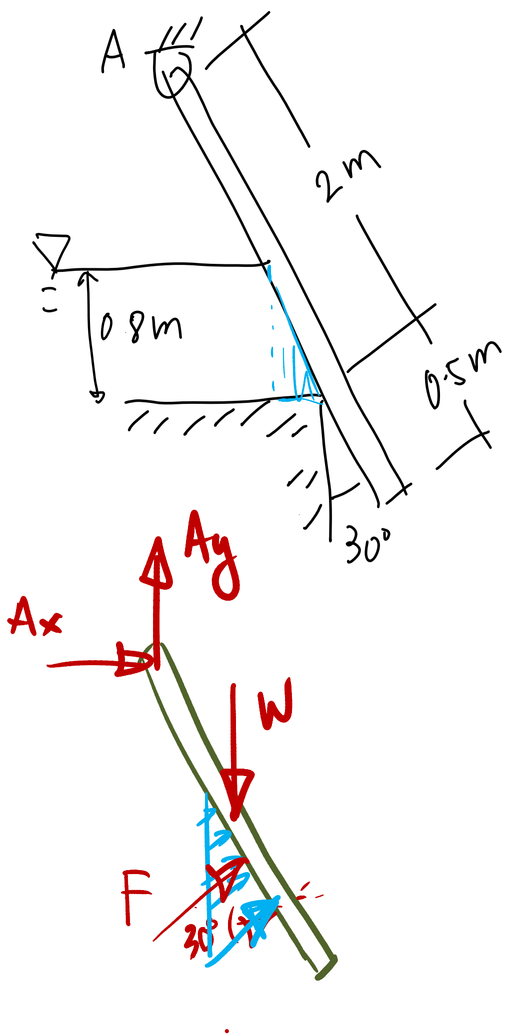
$$F_2 = w_c\, 2m = 392.4\ kN \qquad d_2 = 1\,m$$

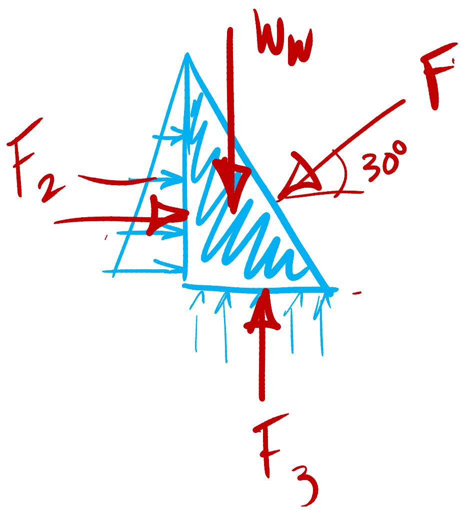
$$\Sigma F_y = 0 = A_y - F_2 \Rightarrow A_y = 392.4\ kN$$

$$\Sigma F_x = 0 = A_x + B_x - F_4 \Rightarrow B_x = 359.7\ kN$$

$$\Sigma M_B = 0 = -F_2 d_2 - F_4 d_4 + A_x(4m) + A_y(2m) \Rightarrow A_x = \frac{1700.4}{4} = 425.1\ kN$$

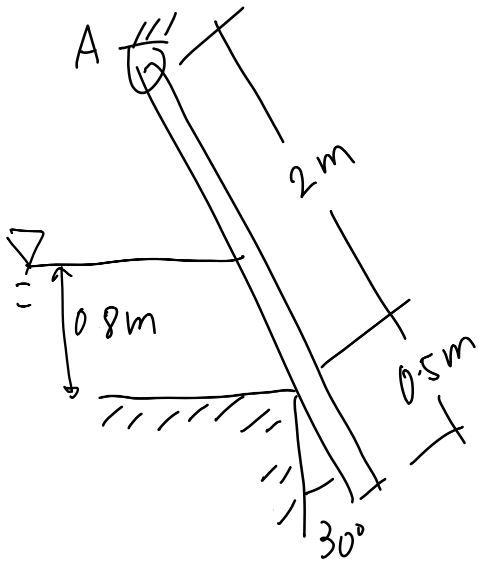
Example: The fresh water channel is contained by a 4-m wide plate hanging freely hinged at A. If the gate is designed to open when water reaches 0.8 m, what must be the weight W of the gate.





Ans: W = 39,250 N

Example: The fresh water channel is contained by a 4-m wide plate hanging freely hinged at A. If the gate is designed to open when water reaches 0.8 m, what must be the weight W of the gate.



Ans: W = 39,250 N