MOMENTS OF INERTIA FOR AREAS, RADIUS OF GYRATION OF AN AREA, & MOMENTS OF INTERTIA BY INTEGRATION

<u>Today's Objectives</u>:

Students will be able to:

a) Define the moments of inertia (MoI) for an area.

b) Determine the MoI for an area by integration.

<u>In-Class Activities</u>:

- Check Homework, if any
- Reading Quiz
- Applications
- MoI: Concept and Definition
- MoI by Integration
- Concept Quiz
- Group Problem Solving
- Attention Quiz

READING QUIZ

1. The definition of the Moment of Inertia for an area involves an integral of the form

   A) $\int x \, dA$.
   B) $\int x^2 \, dA$.
   C) $\int x^2 \, dm$.
   D) $\int m \, dA$.

2. Select the SI units for the Moment of Inertia for an area.

   A) m^3
   B) m^4
   C) $kg \cdot m^2$
   D) $kg \cdot m^3$

APPLICATIONS

Many structural members like beams and columns have cross sectional shapes like I, H, C, etc..

Why do they usually not have solid rectangular, square, or circular cross sectional areas?

What primary property of these members influences design decisions?

How can we calculate this property?

APPLICATIONS
(continued)

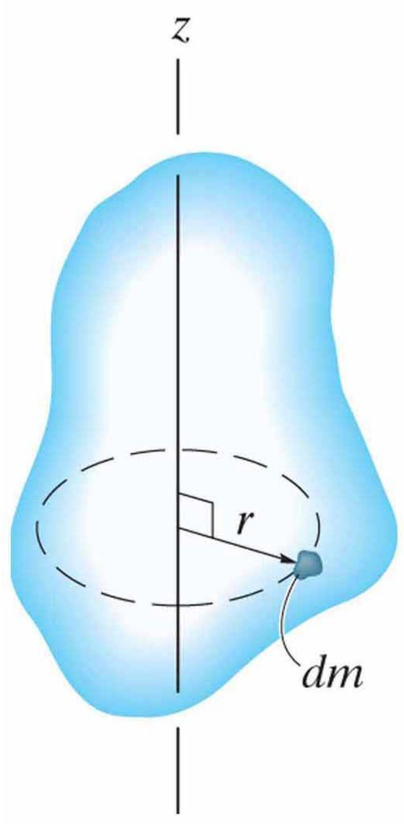
Many structural members are made of tubes rather than solid squares or rounds.

Why?

What parameters of the cross sectional area influence the designer’s selection?

How can we determine the value of these parameters for a given area?

First, consider Mass Moment of Inertia



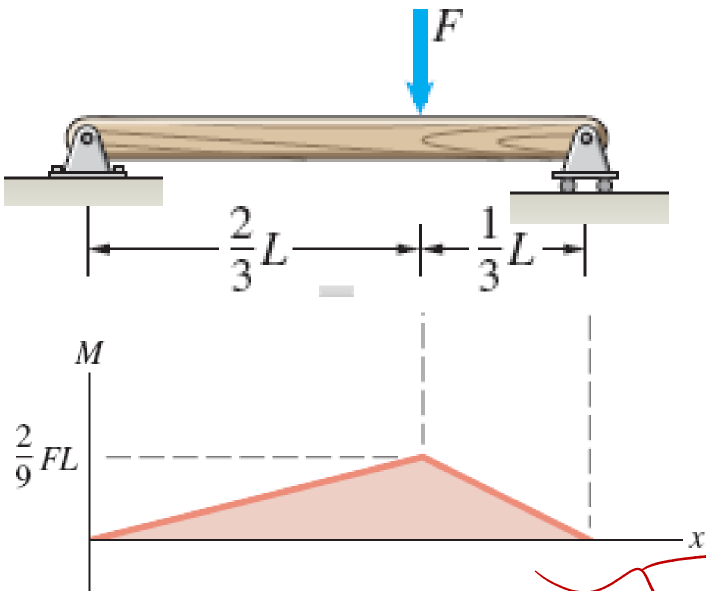
Used for rotational dynamics

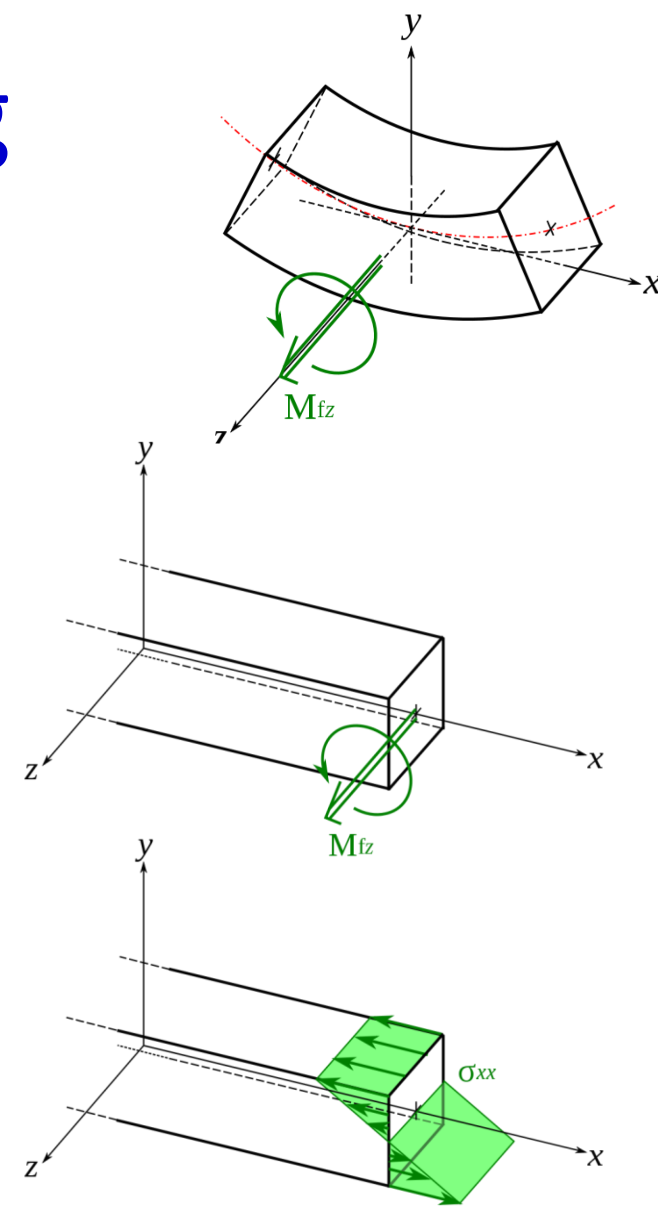
$$M_z = I_z\alpha \qquad K_{rot} = I_z\omega^2$$

$$I_z = \int r^2\,dm$$

$$= \rho \int r^2\,dV \quad \text{for const } \rho$$

Beam Bending





Normal Stress: $\sigma = \frac{My}{I}$

Elastic Strain: $\epsilon = {}^{\sigma}/_{E}$

Deflection of Beam: $\frac{d^2v}{dx^2} = \frac{M}{EI}$

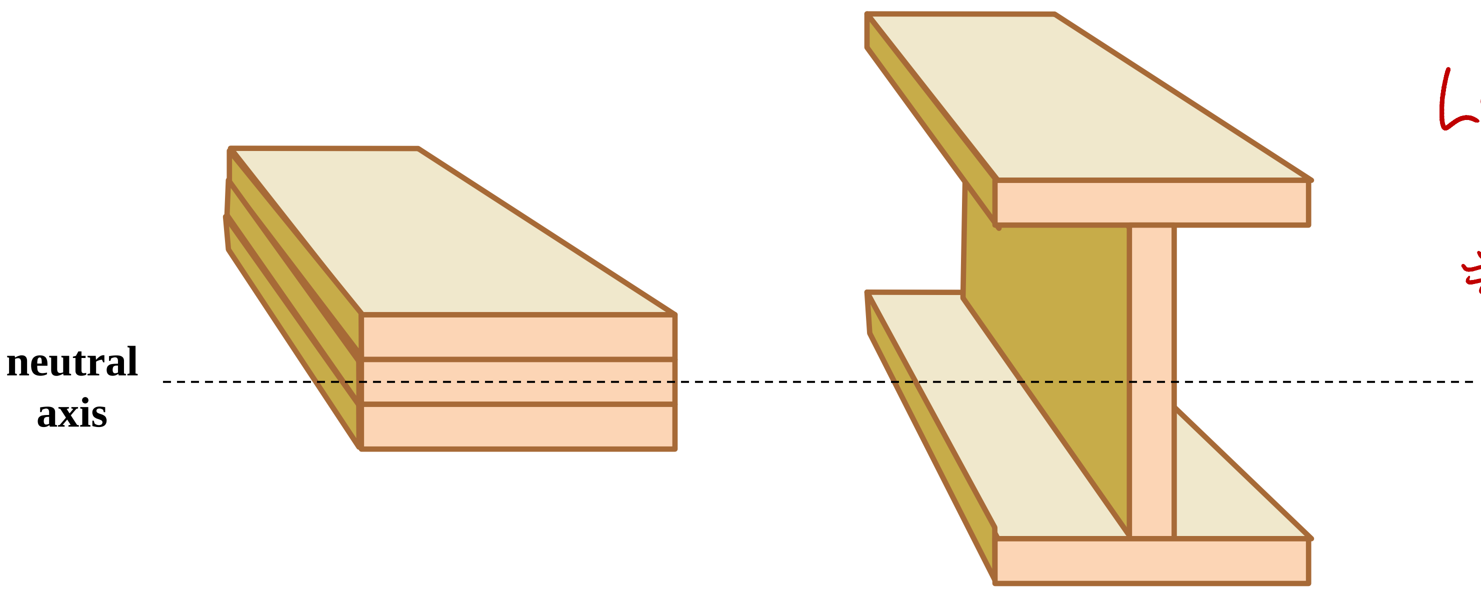
Effect of I on Bending

Normal Stress

$$\sigma = \frac{My}{I}$$

Deflection of Beam

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

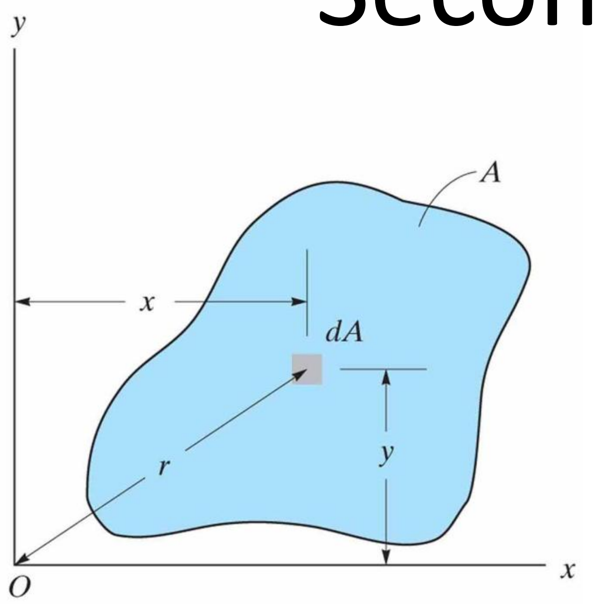


Larger I

⇒ Smaller σ

Stiffer Beam

Second Moment of Area

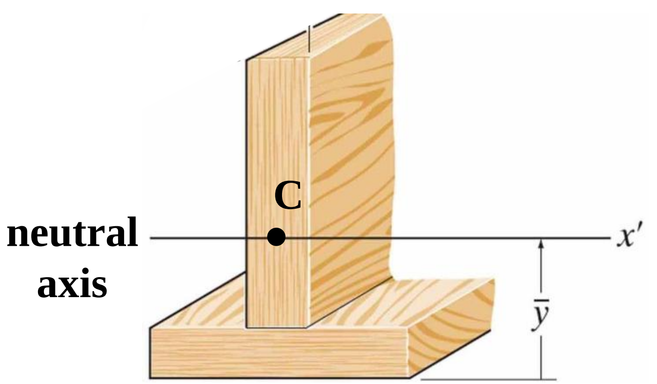


Definitions:

$$I_x = \int y^2 \, dA$$

$$I_y = \int x^2 \, dA$$

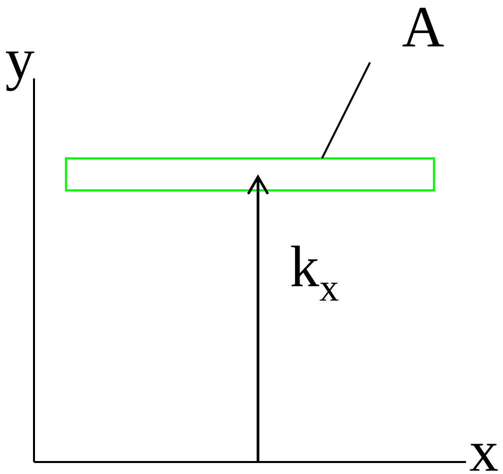
$$J_o = \int r^2 \, dA = I_x + I_y$$



$$A = \int y^0 \, dA \qquad 0^{th} \text{ moment}$$

$$\bar{y} A = \int y \, dA \qquad 1^{st} \text{ moment}$$

RADIUS OF GYRATION OF AN AREA
(Section 10.3)



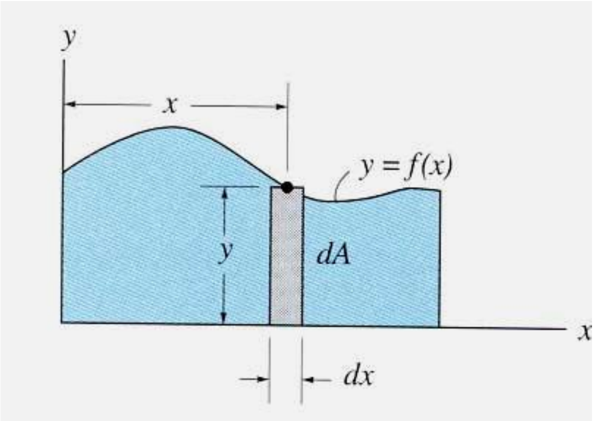
For a given area A and its MoI, I_x , imagine that the entire area is located at distance k_x from the x axis.

Then, $I_x = k_x^2 A$ or $k_x = \sqrt{(I_x / A)}$. This k_x is called the radius of gyration of the area about the x axis. Similarly;

$$k_Y = \sqrt{(I_y / A)} \quad \text{and} \quad k_O = \sqrt{(J_O / A)}$$

The radius of gyration has units of length and gives an indication of the spread of the area from the axes. This characteristic is important when designing columns.

MoI FOR AN AREA BY INTEGRATION (Section 10.4)



For simplicity, the area element used has a differential size in only one direction (dx or dy). This results in a single integration and is usually simpler than doing a double integration with two differentials, dx·dy.

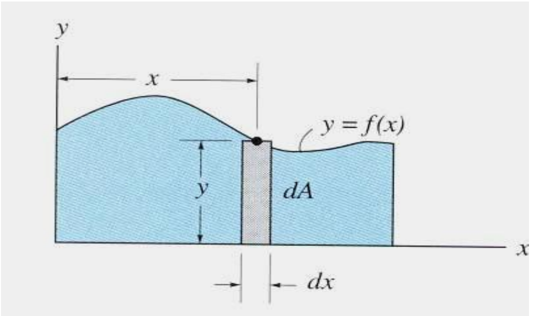
The step-by-step procedure is:

1. Choose the element dA: There are two choices: a vertical strip or a horizontal strip. Some considerations about this choice are:

   a) The element parallel to the axis about which the MoI is to be determined usually results in an easier solution. For example, we typically choose a horizontal strip for determining I_x and a vertical strip for determining I_y.

MoI FOR AN AREA BY INTEGRATION

(continued)



b) If y is easily expressed in terms of x (e.g., $y = x^2 + 1$), then choosing a vertical strip with a differential element dx wide may be advantageous.

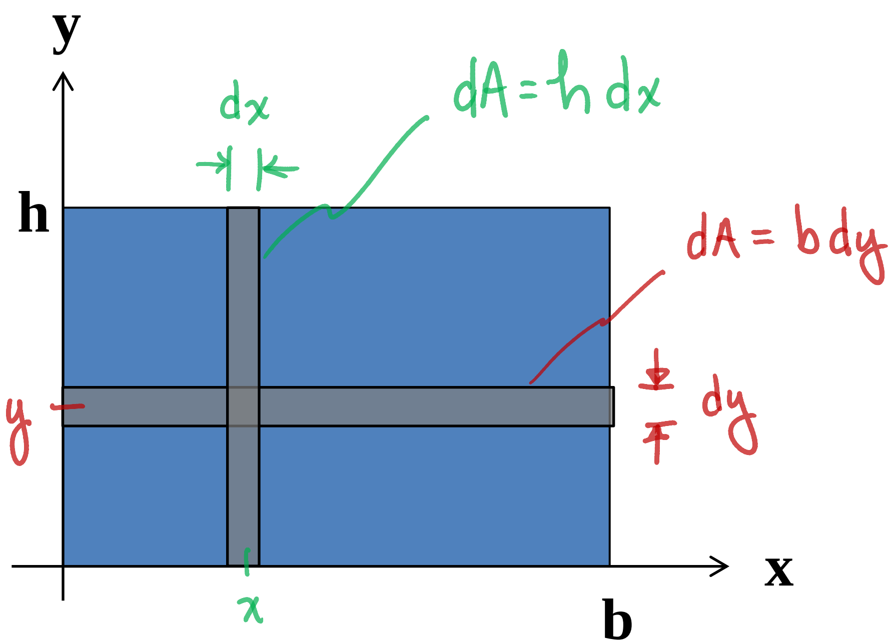
2. Integrate to find the MoI. For example, given the element shown in the figure above:

$I_y = \int x^2 \, dA = \int x^2 \, y \, dx$ and

$I_x = \int dI_x = \int (1/3)\, y^3 \, dx$ (using the information for a rectangle about its base from the inside back cover of the textbook).

Since in this case the differential element is dx, y needs to be expressed in terms of x and the integral limit must also be in terms of x. As you can see, choosing the element and integrating can be challenging. It may require a trial and error approach plus experience.

Moment by Integration Example: Find I_x & I_y



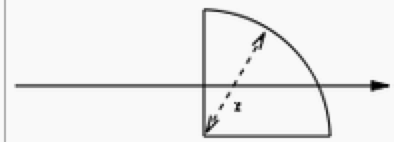
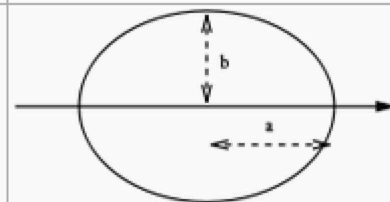
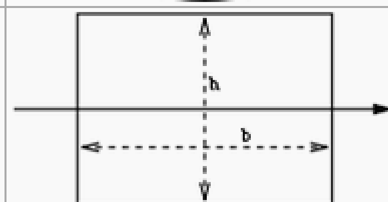
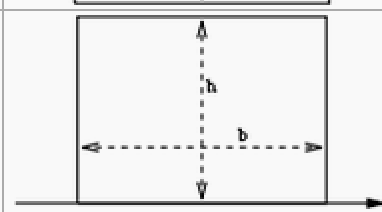
$$I_x = \int y^2 \, dA$$

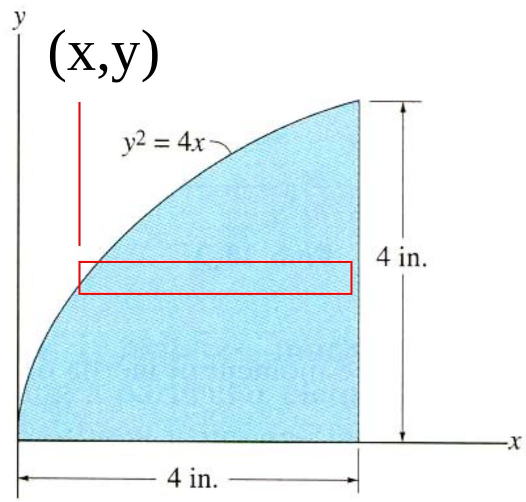
$$= \int_0^h y^2 \, b \, dy = b \left(\frac{y^3}{3} \right)_0^h$$

$$\Rightarrow \quad I_x = \frac{1}{3} b h^3$$

$$I_y = \int x^2 \, dA = \int_0^b x^2 \, h \, dx = h \left(\frac{x^3}{3} \right)_0^b \Rightarrow I_y = \frac{1}{3} b^3 h$$

http://en.wikipedia.org/wiki/List_of_area_moments_of_inertia

a filled quarter circle with radius *r* entirely in the 1st quadrant of the Cartesian coordinate system		$I = \frac{\pi r^4}{16}$
a filled quarter circle as above but with respect to a horizontal or vertical axis through the centroid		$I_0 = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$
a filled ellipse whose radius along the *x*-axis is *a* and whose radius along the *y*-axis is *b*		$I_0 = \frac{\pi}{4} ab^3$
a filled rectangular area with a base width of *b* and height *h*		$I_0 = \frac{bh^3}{12}$
a filled rectangular area as above but with respect to an axis collinear with the base		$I = \frac{bh^3}{3}$



EXAMPLE

Given: The shaded area shown in the figure.

Find: The MoI of the area about the x- and y-axes.

<u>Plan</u>: Follow the steps given earlier.

<u>Solve for Ix</u>

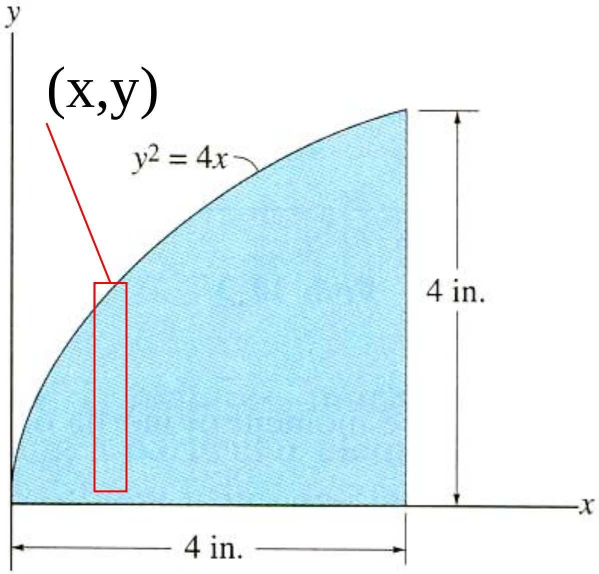
$$I_x = \int y^2 \, dA$$

$$dA = (4 - x)\, dy = (4 - y^2/4)\, dy$$

$$I_x = \int_O^4 y^2 \, (4 - y^2/4)\, dy$$

$$= \left[\, (4/3)\, y^3 - (1/20)\, y^5 \,\right]_0^4 = 34.1 \text{ in}^4$$

EXAMPLE
(continued)



$$I_y = \int x^2 \; dA = \int x^2 \; y \; dx$$

$$= \int x^2 \; (2 \sqrt{x}) \, dx$$

$$= 2 \; {}_0\!\int^4 x^{2.5} \; dx$$

$$= \left[(2/3.5) \;\; x^{3.5} \right]_0^4$$

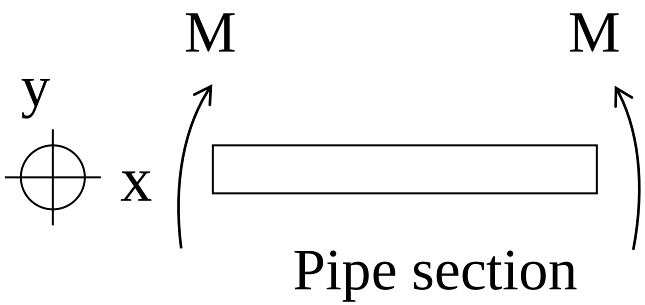
$$= \; 73.1 \;\; in^4$$

In the above example, it will be difficult to determine I_y using a horizontal strip. However, I_x in this example can be determined using a vertical strip. So,

$I_x = \int (1/3) \, y^3 \, dx = \int (1/3) \, (2\sqrt{x})^3 \, dx$.

CONCEPT QUIZ

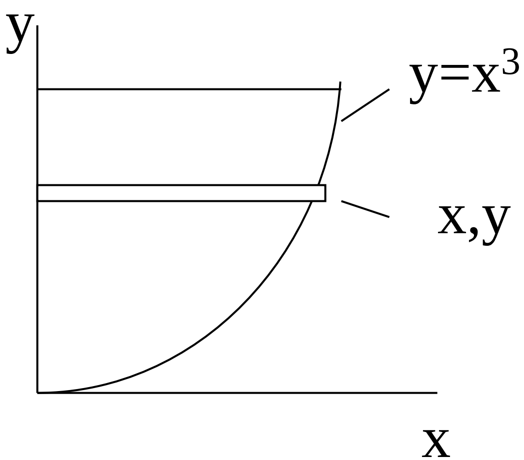
1. A pipe is subjected to a bending moment as shown. Which property of the pipe will result in lower stress (assuming a constant cross-sectional area)?



A) Smaller I_x B) Smaller I_y

C) Larger I_x D) Larger I_y

2. In the figure to the right, what is the differential moment of inertia of the element with respect to the y-axis (dI_y)?



A) x^2 y dx B) (1/12) x^3 dy

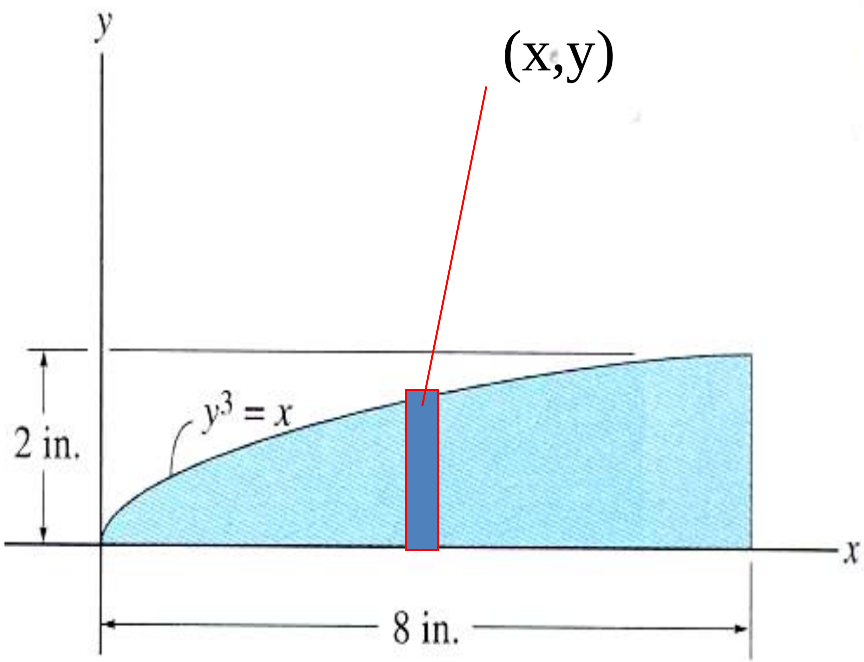
C) y^2 x dy D) (1/3) y dy

GROUP PROBLEM SOLVING

Given: The shaded area shown.

Find: I_x and I_y of the area.

Plan: Follow the steps described earlier.



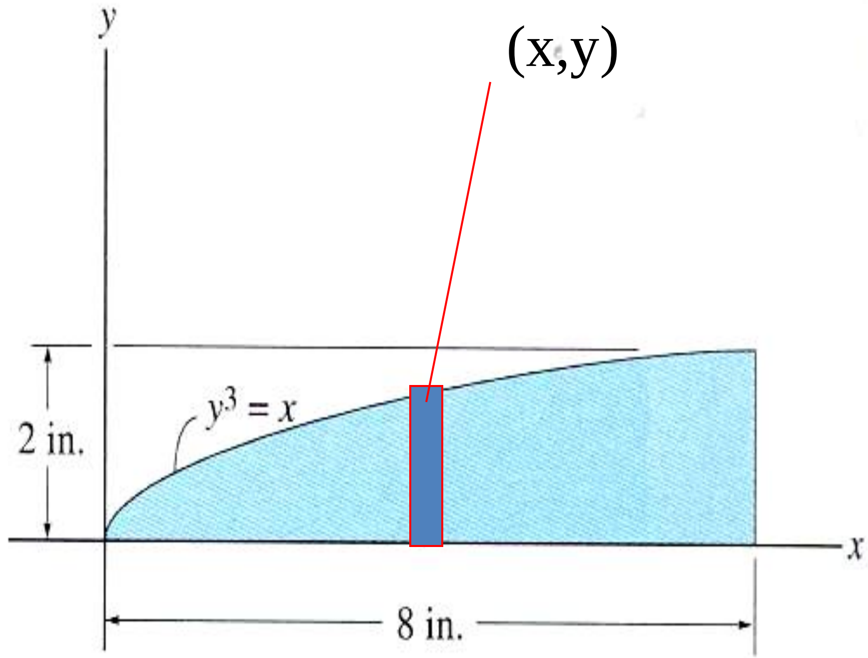
Solution

$$I_x = \int (1/3)\ y^3 \ \ dx$$

$$= \int_0^8 (1/3)\, x \ dx \quad = \left[x^2 / 6\ \right]_0^8$$

$$= \ 10.7 \ \text{in}^4$$

GROUP PROBLEM SOLVING
(continued)



$$I_Y = \int x^2 \, dA = \int x^2 \, y \, dx$$

$$= \int x^2 \, (x^{(1/3)}) \, dx$$

$$= {}_0\int^8 x^{(7/3)} \, dx$$

$$= [(3/10) \, x^{(10/3)}]_0^8$$

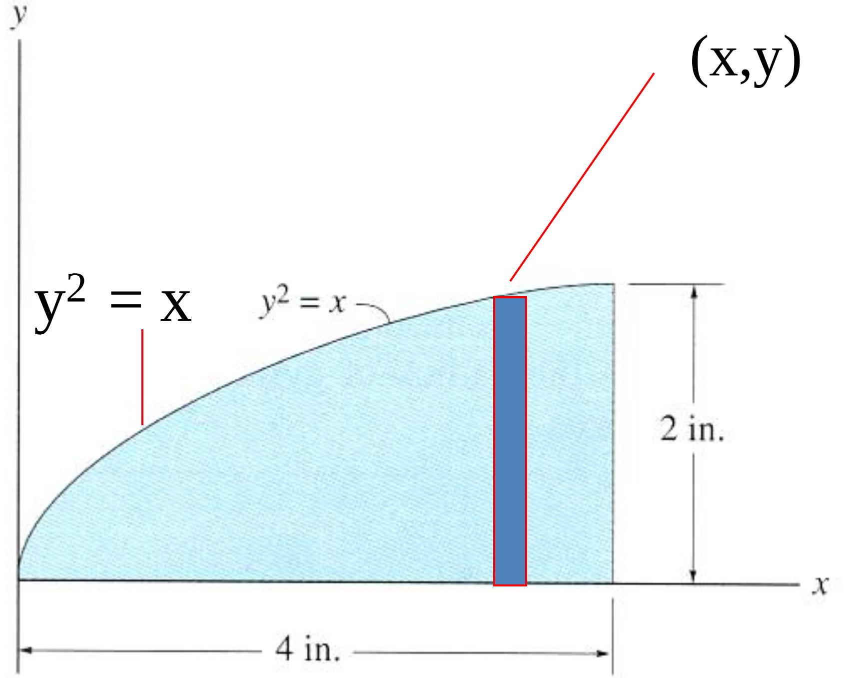
$$= 307 \quad in^4$$

ATTENTION QUIZ

1. When determining the MoI of the element in the figure, dI_y equals

   A) $x^2\ dy$ B) $x^2\ dx$

   C) $(1/3)\ y^3\ dx$ D) $x^{2.5}\ dx$

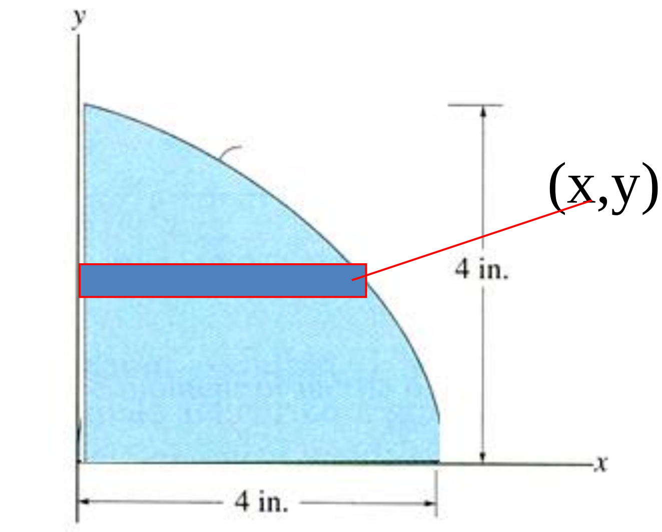


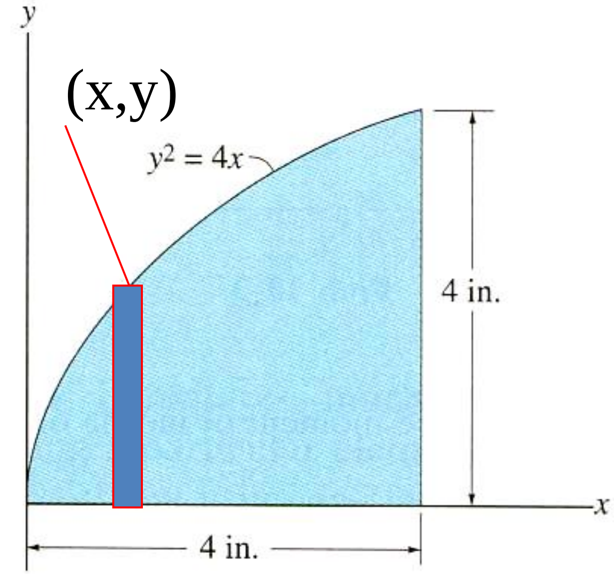
2. Similarly, dI_x equals

   A) $(1/3)\ x^{1.5}\ dx$ B) $y^2\ dA$

   C) $(1/12)\ x^3\ dy$ D) $(1/3)\ x^3\ dx$

Summary of MoI calculation





Horizontal Strip

$$I_x = \int y^2 \, dA$$

if base of strip is on y-axis

$$I_y = \int (1/3) \, x^3 \, dy$$

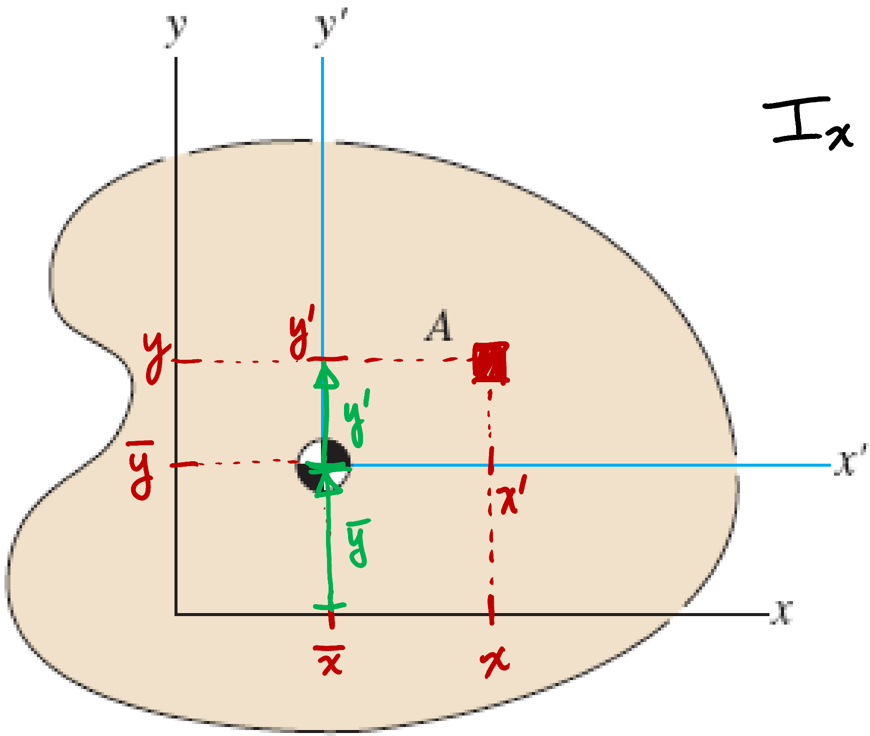
Vertical Strip

$$I_y = \int x^2 \, dA$$

if base of strip is on x-axis

$$I_x = \int (1/3) \, y^3 \, dx$$

Parallel-Axis Theorem



$$I_x = \int y^2 dA = \int (y' + \bar{y})^2 dA$$

$$= \int y'^2 + 2y'\bar{y} + \bar{y}^2 dA$$

$$= \underbrace{\int y'^2 dA}_{\bar{I}_{x'}} + 2\bar{y}\underbrace{\int y' dA}_{= \bar{y}' = 0} + \bar{y}^2 \underbrace{\int dA}_{A}$$

$$I_x = \bar{I}_{x'} + \bar{y}^2 A$$

$$I_y = \bar{I}_{y'} + \bar{x}^2 A$$

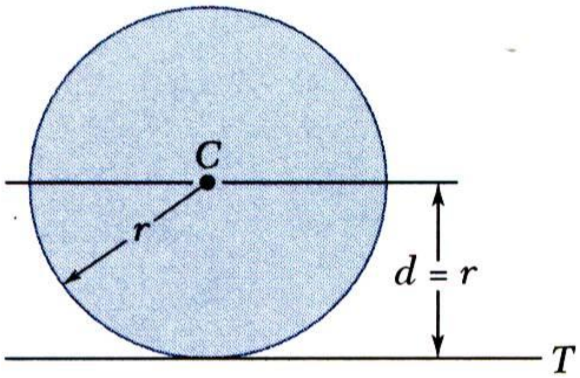
moments in non-centroidal x, y system

centroidal moments

centroid location in x, y

Parallel Axis Theorem Examples

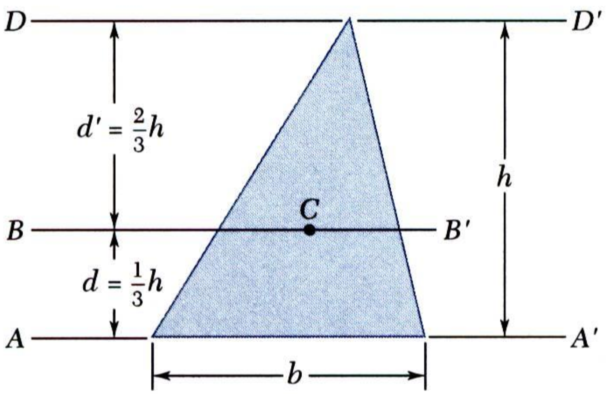
$$\bar{I} = \tfrac{1}{4}\pi r^4$$



- Moment of inertia I_T of a circular area with respect to a tangent to the circle:

$$I_T = \bar{I} + \bar{y}^2 A$$

$$= \tfrac{1}{4}\pi r^4 + r^2(\pi r^2) = \underline{\underline{\tfrac{5}{4}\pi r^4}}$$



$$I_{AA'} = \tfrac{1}{12} b h^3$$

- Moment of inertia of a triangle with respect to a centroidal axis:

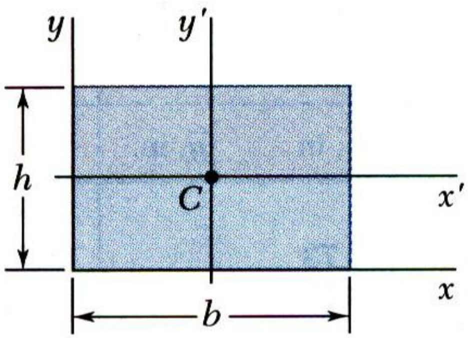
$$I_{AA'} = \bar{I}_{BB'} + \bar{y}^2 A$$

$$\Rightarrow \bar{I}_{BB'} = I_{AA'} - \bar{y}^2 A$$

$$= \tfrac{1}{12} b h^3 - \left(\tfrac{1}{3}h\right)^2\left(\tfrac{1}{2}bh\right) = bh^3\left(\tfrac{1}{12} - \tfrac{1}{18}\right)$$

$$\Rightarrow \bar{I}_{BB'} = \tfrac{1}{36} b h^3$$

Moments of Inertia of Common Areas



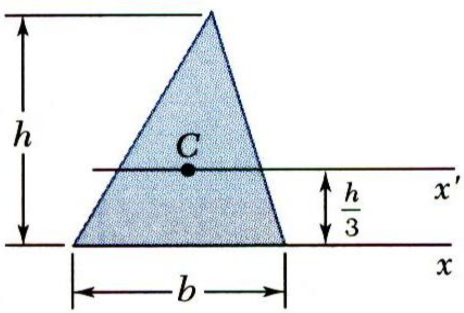
$$\bar{I}_{x'} = \frac{1}{12}bh^3$$

$$\bar{I}_{y'} = \frac{1}{12}b^3h$$

$$I_x = \frac{1}{3}bh^3$$

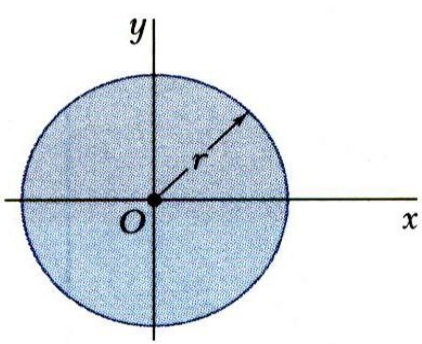
$$I_y = \frac{1}{3}b^3h$$

$$J_C = \frac{1}{12}bh(b^2 + h^2)$$



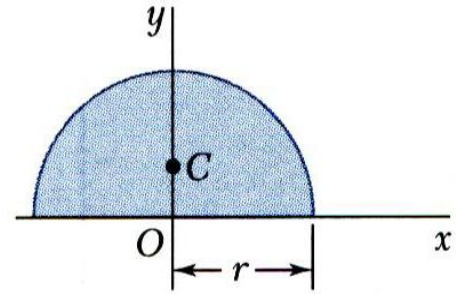
$$\bar{I}_{x'} = \frac{1}{36}bh^3$$

$$I_x = \frac{1}{12}bh^3$$



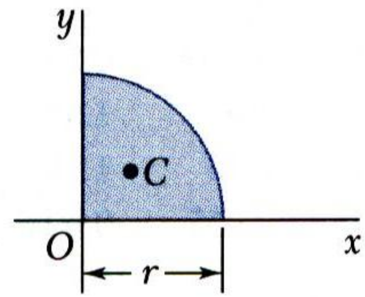
$$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$$

$$J_O = \frac{1}{2}\pi r^4$$



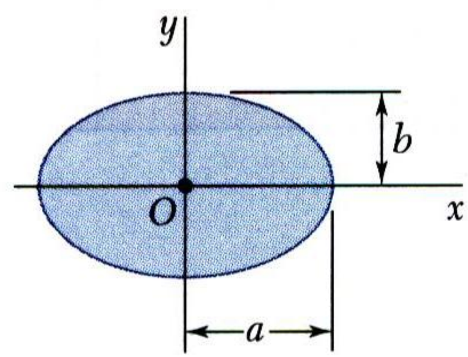
$$I_x = I_y = \frac{1}{8}\pi r^4$$

$$J_O = \frac{1}{4}\pi r^4$$



$$I_x = I_y = \frac{1}{16}\pi r^4$$

$$J_O = \frac{1}{8}\pi r^4$$

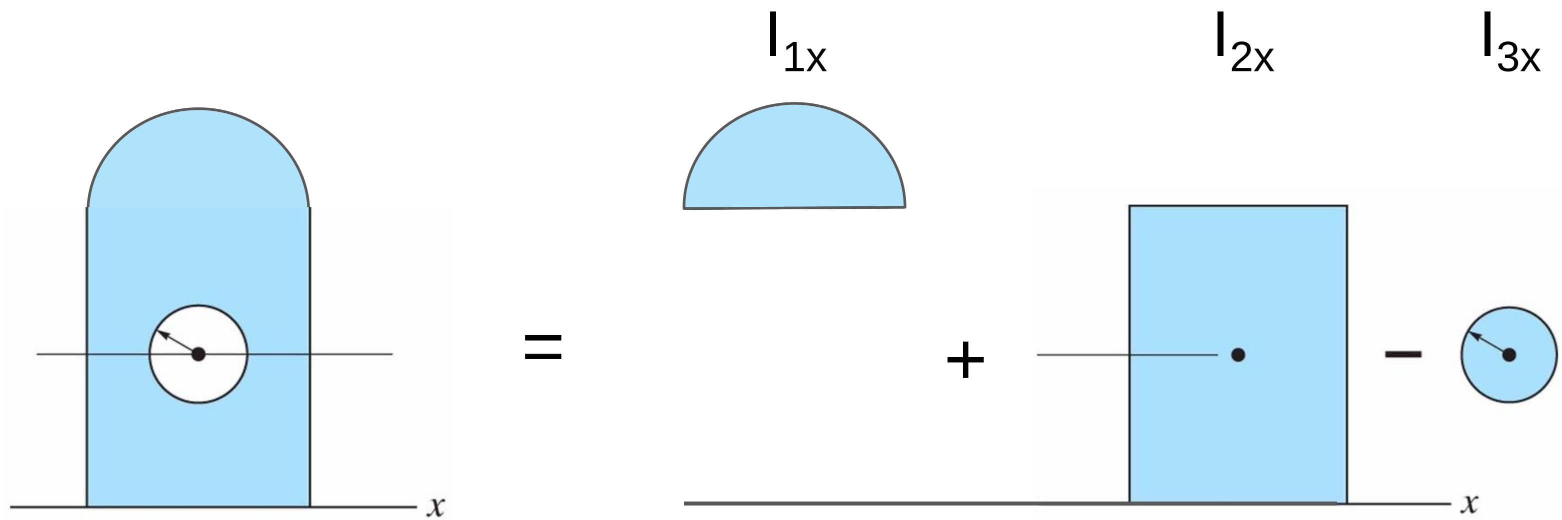


$$\bar{I}_x = \frac{1}{4}\pi ab^3$$

$$\bar{I}_y = \frac{1}{4}\pi a^3b$$

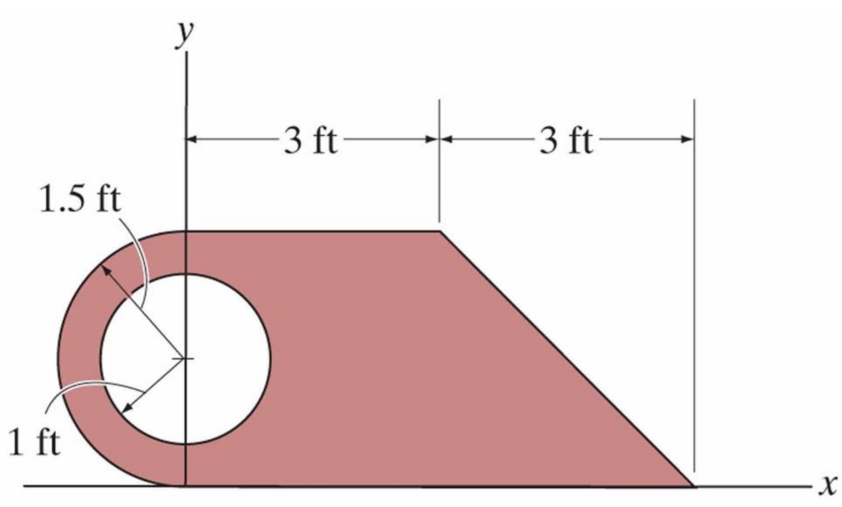
$$J_O = \frac{1}{4}\pi ab(a^2 + b^2)$$

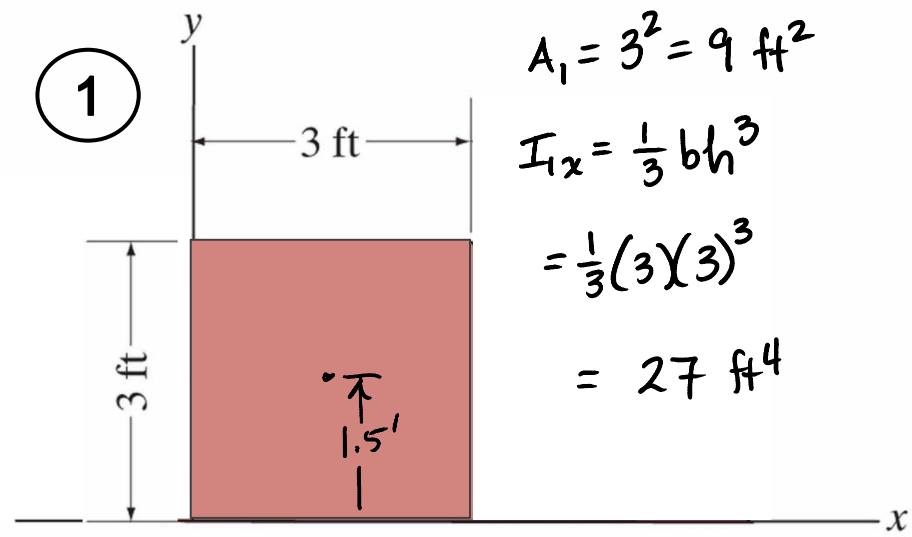
2^{nd} Moments of Composite Areas



$$I_x = I_{1x} + I_{2x} - I_{3x}$$

Composite Area Example: Find I_x





$A_1 = 3^2 = 9 \text{ ft}^2$

$I_{1x} = \frac{1}{3} b h^3$

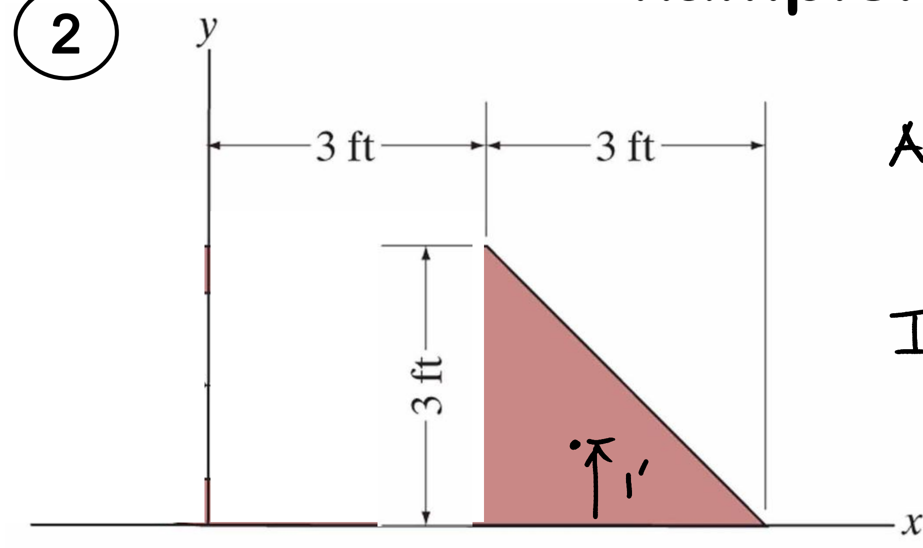
$= \frac{1}{3}(3)(3)^3$

$= 27 \text{ ft}^4$

#	A_i (ft²)	$\tilde{y}_i$ (ft)	$\bar{I}_{ix'}$ (ft⁴)	I_{ix} (ft⁴)
1	9	1.5		27
2				
3				
4				

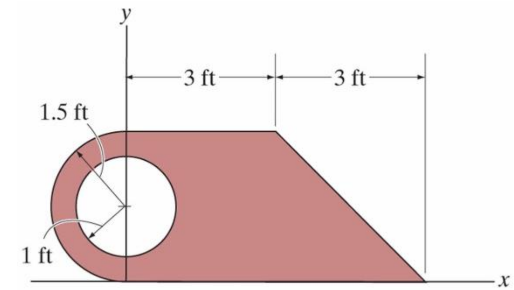
Example: Find I_x

2



$A_2 = \frac{1}{2} bh$
$= 4.5 \text{ ft}^2$

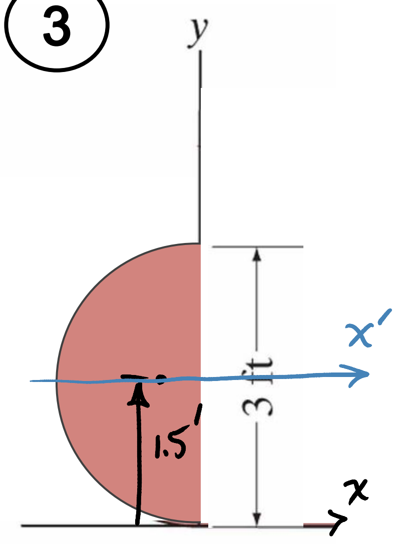
$I_{2x} = \frac{1}{12} bh^3 = \frac{1}{12}(3)(3)^3 = 6.75$

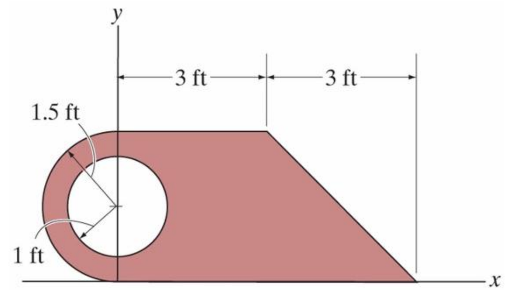


#	A_i (ft^2)	$\tilde{y}_i$ (ft)	$\bar{I}_{ix'}$ (ft^4)	I_{ix} (ft^4)
1	9	1.5		27
2	4.5	1		6.75
3				
4				

Example: Find I_x

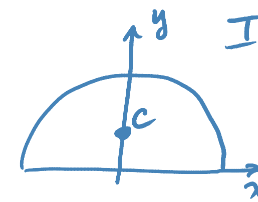
3





$$A_3 = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(1.5)^2 = 3.53$$

$$\bar{I}_{3x'} = \frac{1}{8}\pi r^4 = \frac{1}{8}\pi(1.5)^4 = 1.99$$



$I_y = \frac{1}{8}\pi r^4$

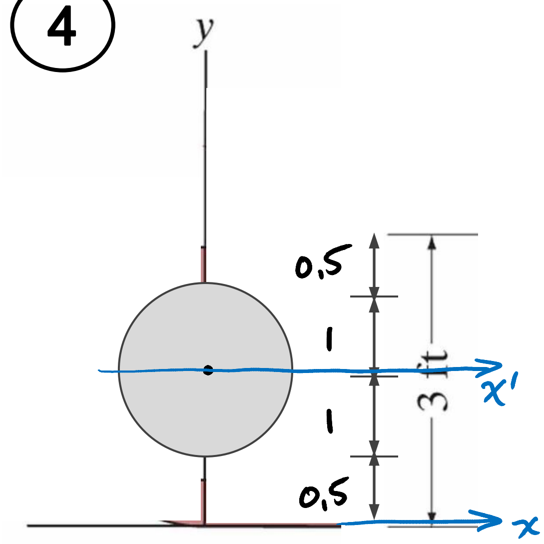
$$I_{3x} = \bar{I}_{3x'} + \bar{y}_3^2 A_3$$

$$= 1.99 + (1.5)^2(3.53) = 9.93\ ft^4$$

#	A_i (ft²)	$\tilde{y}_i$ (ft)	$\bar{I}_{ix'}$ (ft⁴)	I_{ix} (ft⁴)
1	9	1.5		27
2	4.5	1		6.75
3	3.53	1.5	1.99	9.93
4				

Example: Find I_x

4



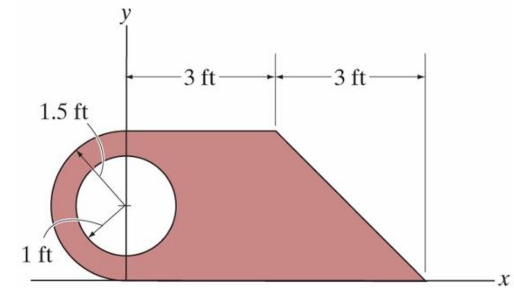
$$\bar{y}_4 = 1.5'$$

$$A_4 = -\pi r^2 = -\pi(1)^2$$

$$\bar{I}_{4x'} = -\frac{1}{4}\pi r^4 = -\frac{1}{4}\pi(1)^4 = -0.785$$

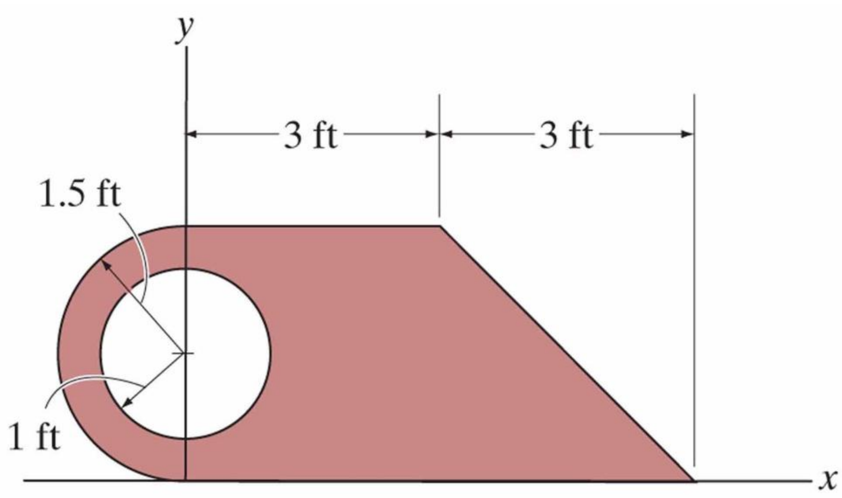
$$I_{4x} = \bar{I}_{4x'} + \bar{y}_4^{\,2} A_4$$

$$= -0.785 + (1.5)^2(-3.14) = -7.85$$



#	A_i (ft^2)	$\tilde{y}_i$ (ft)	$\bar{I}_{ix'}$ (ft^4)	I_{ix} (ft^4)
1	9	1.5		27
2	4.5	1		6.75
3	3.53	1.5	1.99	9.93
4	−3.14	1.5	−0.785	−7.85

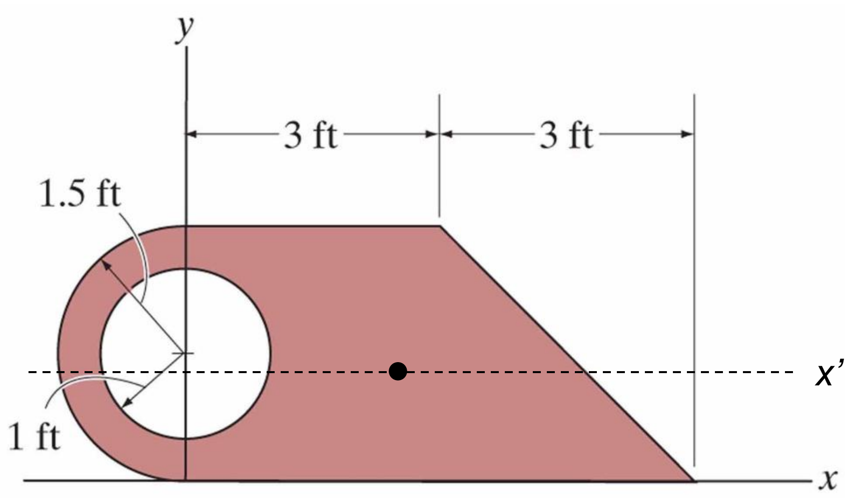
Example: Find I_x



$I_x = \sum I_{ix}$

#	A_i (ft²)	$\tilde{y}_i$ (ft)	$\bar{I}_{ix'}$ (ft⁴)	I_{ix} (ft⁴)
1	9	1.5		27
2	4.5	1		6.75
3	3.53	1.5	1.99	9.93
4	–3.14	1.5	–.785	–7.85

Example: Find $\bar{I}_{x'}$ about the neutral axis



$$I_x = \bar{I}_{x'} + \bar{y}^2 A$$

$$\Rightarrow \bar{I}_{x'} = 35.83 - (1.34)^2(13.89)$$

$$\bar{I}_{x'} = 10.89 \text{ ft}^4$$

*y_c = 1.34 from centroid analysis in earlier lecture

#	A_i (ft²)	$\tilde{y}_i$ (ft)	$\bar{I}_{ix'}$ (ft⁴)	I_{ix} (ft⁴)
1	9	1.5		27
2	4.5	1		6.75
3	3.53	1.5	1.99	9.93
4	–3.14	1.5	–.785	–7.85

35.83

End of the Lecture

Let Learning Continue