Circuit Theory

Chapter 8
Second-Order Circuits

Second-Order Circuits
Chapter 8

8.1 Examples of 2nd order RLC circuit

8.2 The source-free series RLC circuit

8.3 The source-free parallel RLC circuit

8.4 Step response of a series RLC circuit

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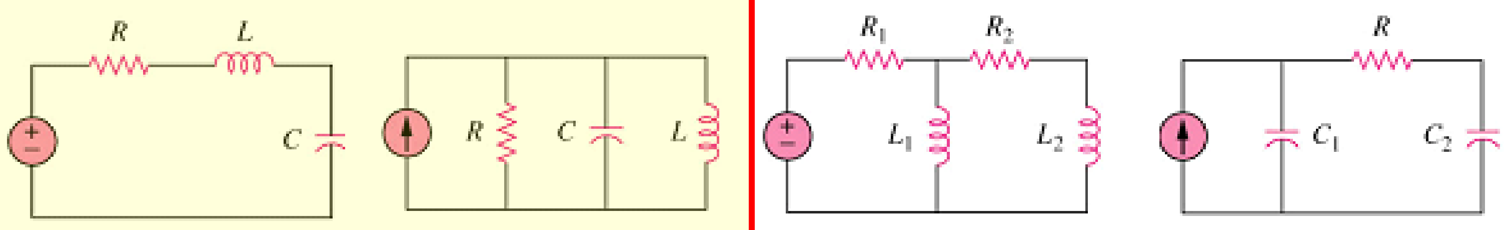
8.1 Examples of Second Order RLC circuits

What is a 2nd order circuit?

A second-order circuit is characterized by a <u>second-order differential equation</u>. It consists of <u>resistors</u> and the equivalent of <u>two energy storage elements</u>.



(a) (b) (c) (d)

RLC Series RLC Parallel RL T-config RC Pi-config

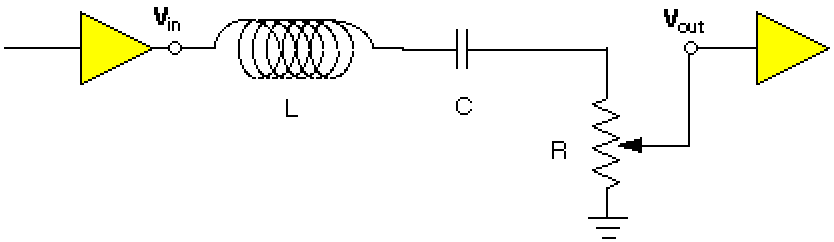
An RLC circuit contains both an inductor and a capacitor

- These circuits have a wide range of applications, including oscillators and frequency filters
- They can also model automobile suspension systems, temperature controllers, airplane responses, etc.

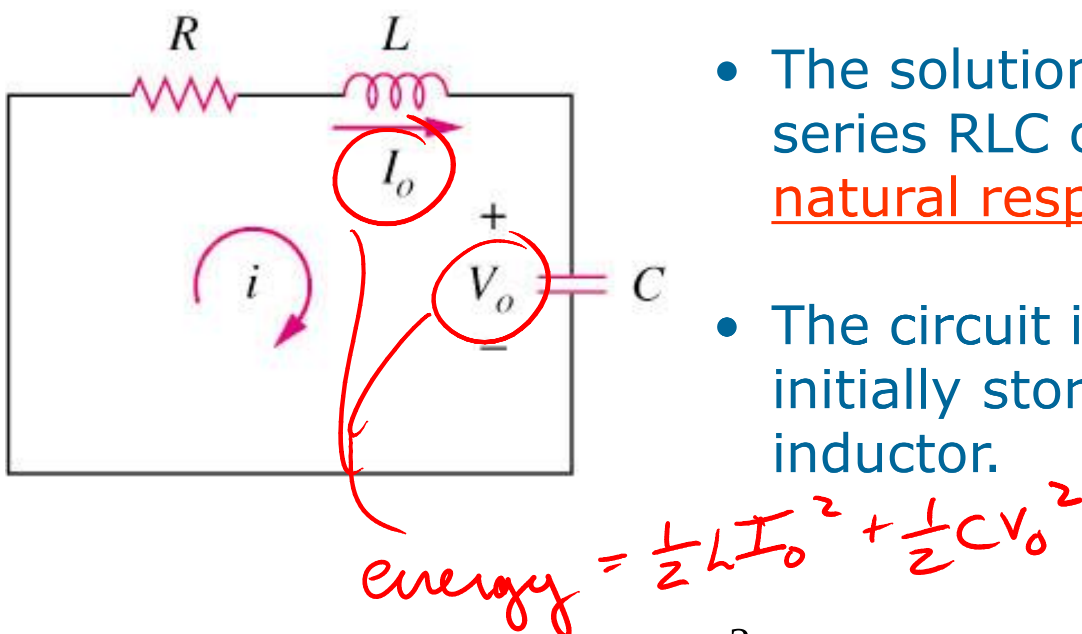
RLC circuits

A very common use of inductors is as part of a **tuned circuit**. For instance, each slider on a graphic equalizer selects out a range of frequencies from the audio spectrum, and controls how much gain there should be for those frequencies. A later stage recombines the various frequency bands.

RLC (Resistor-Inductor-Capacitor) circuits are a common way of extracting a frequency range. Here's part of a typical circuit:



8.2 Source-Free **Series** RLC Circuits (1)



- The solution of the source-free series RLC circuit is called the natural response of the circuit.
- The circuit is excited by the energy initially stored in the capacitor and inductor.

The 2nd order of expression

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

How to derive and how to solve?

Derivation

Series RLC circuit

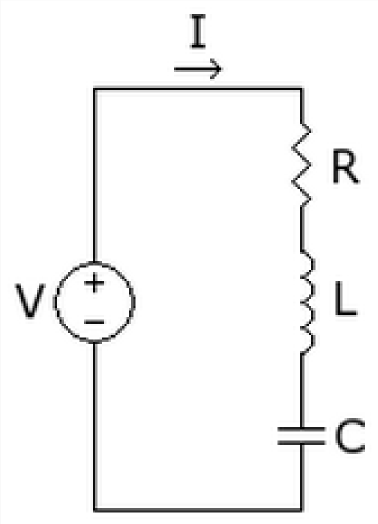


Figure 1. RLC series circuit

- **V** - the voltage of the power source
- **I** - the current in the circuit
- **R** - the resistance of the resistor
- **L** - the inductance of the inductor
- **C** - the capacitance of the capacitor

From KVL,

$$v_R + v_L + v_C = v(t)$$

where v_R, v_L, v_C are the voltages across R, L and C respectively and $v(t)$ is the time varying voltage from the source. Substituting in the constitutive equations,

$$Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{\tau=t} i(\tau)\, d\tau = v(t)$$

For the case where the source is an unchanging voltage, differentiating and dividing by L leads to the second order differential equation:

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0$$

This can usefully be expressed in a more generally applicable form:

$$\frac{d^2 i(t)}{dt^2} + 2\alpha\frac{di(t)}{dt} + {\omega_0}^2 i(t) = 0$$

8.2 Source-Free Series RLC Circuits (3)

There are three possible solutions for the following 2nd order differential equation:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

=> $$\frac{d^2 i}{dt^2} + 2\alpha\frac{di}{dt} + \omega_0^2 i = 0$$

General 2nd order Form

Natural or Resonant Freq

Neper Freq

where $$\alpha = \frac{R}{2L} \quad and \quad \omega_0 = \sqrt{\frac{1}{LC}}$$

The types of solutions for i(t) depend on the relative values of α and ω.

To solve (1) $\frac{d^2 i}{dt^2} + 2\alpha \frac{di}{dt} + \omega_0^2 i = 0$

- Assume a solution of the form

$$i(t) = Ae^{st}$$

- Then $\frac{di}{dt} = sAe^{st}$ and $\frac{d^2 i}{dt^2} = s^2 Ae^{st}$
- Substitute into (1), $s^2 Ae^{st} + 2\alpha sAe^{st} + \omega_0^2 Ae^{st} = 0$
- And divide out the common terms

  to get $s^2 + 2\alpha s + \omega_0^2 = 0$ (the characteristic eqn)
- Solve for s: $s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

- s has either 2 real, 1 real, or 2 complex solutions

8.2 Source-Free Series RLC Circuits (4)

There are three possible solutions for the following 2nd order differential equation:

$$\frac{d^2i}{dt^2}+2\alpha\frac{di}{dt}+\omega_0^2 i=0$$

$$\alpha=\frac{R}{2L} \quad \textit{and} \quad \omega_0=\sqrt{\frac{1}{LC}}$$

1. If $\alpha > \omega_o$, over-damped case

$$i(t)=A_1e^{s_1t}+A_2e^{s_2t}$$ **where** $$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$$

2. If $\alpha = \omega_o$, critical damped case

$$i(t)=(A_2+A_1t)e^{-\alpha t}$$ **where** $$s_{1,2}=-\alpha$$

3. If $\alpha < \omega_o$, under-damped case

$$i(t)=e^{-\alpha t}(B_1\cos\omega_d t+B_2\sin\omega_d t)$$ **where** $$\omega_d=\sqrt{\omega_0^2-\alpha^2}$$

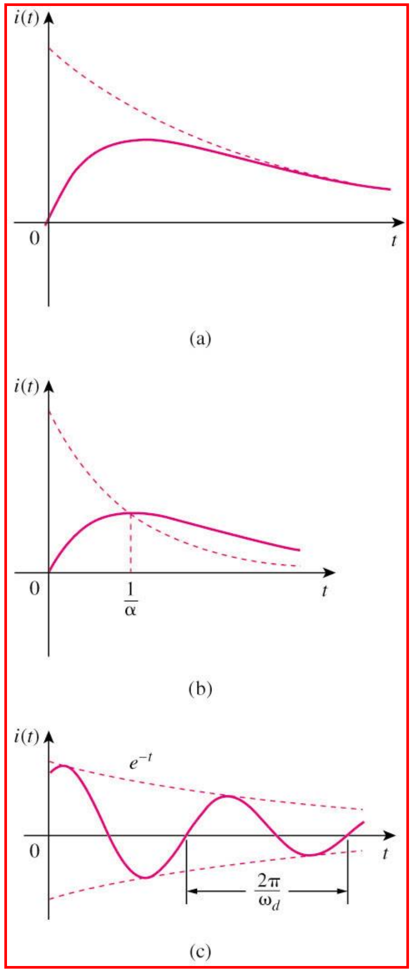
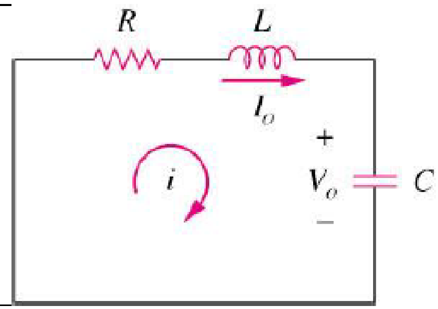
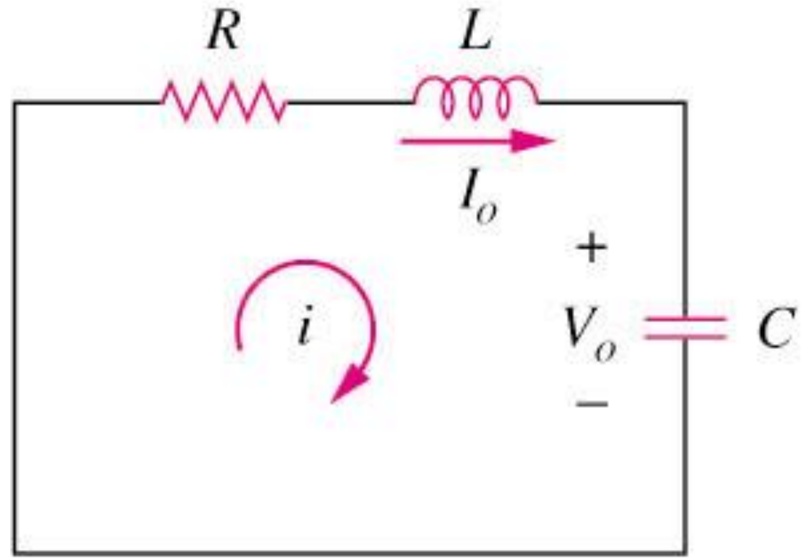


Table 1: Natural Response of Series RLC

Circuit	 		
Diff-Eq	$\frac{d^2i}{dt^2}+2\alpha\frac{di}{dt}+\omega_0^2 i=0$	$\alpha=\frac{R}{2L}$ and $\omega_0=\sqrt{\frac{1}{LC}}$	
Damping	Over: $\omega_0^2 < \alpha^2$	Under: $\omega_0^2 > \alpha^2$	Critical: $\omega_0^2 = \alpha^2$
Form of Soln	$i(t) = A_1e^{s_1t} + A_2e^{s_2t}$	$i(t) = e^{-\alpha t}(B_1\cos\omega_d t + B_2\sin\omega_d t)$	$i(t) = A_1 t\, e^{-\alpha t} + A_2e^{-\alpha t}$
Roots	$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0{}^2}$	$s_{1,2} = -\alpha \pm j\omega_d$, $\omega_d = \sqrt{\omega_0^2-\alpha^2}$	$s_{1,2} = -\alpha$
Boundary Conditions	$i(0+) = A_1 + A_2$ $\frac{di(0+)}{dt} = s_1A_1 + s_2A_2$	$i(0+) = B_1$ $\frac{di(0+)}{dt} = -\alpha B_1 + \omega_d B_2$	$i(0+) = A_2$ $\frac{di(0+)}{dt} = A_1 - \alpha A_2$

1a) Source-Free Series RLC

If R = 10 Ω, L = 5 H, and C = 2 mF, find α, ω_o, s1 and s2.



What type of natural response will the circuit have?

$$\alpha = \frac{R}{2L} = \frac{10}{2(5)} = 1 \text{ Np/s}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(5)(.002)}} = \frac{1}{\sqrt{.01}} = 10 \text{ Rad/s}$$

$\omega_0 > \alpha \rightarrow$ underdamped

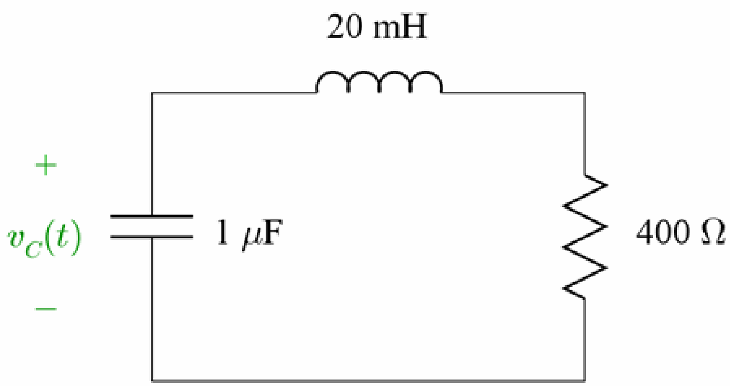
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

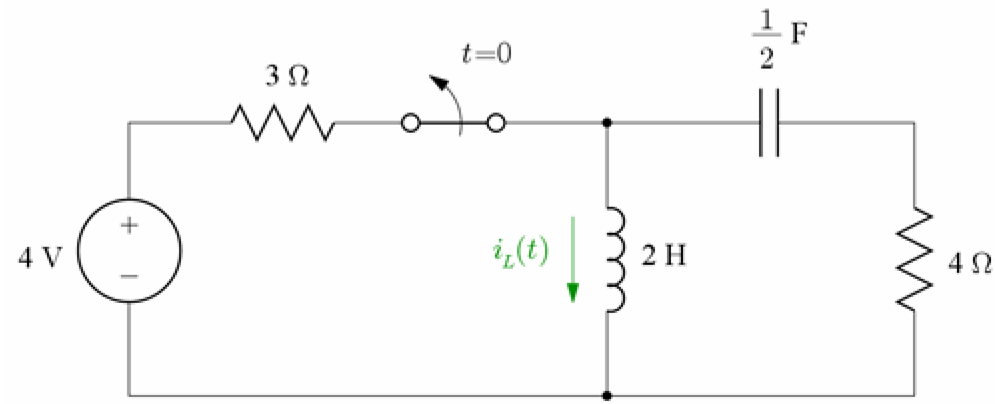
$$s_1 = -1 + \sqrt{1 - 10^2} = -1 + j\,9.95$$

$$s_2 = -1 - j\,9.95$$

Answer: *underdamped*

1b)determine the qualitative form of the response as being either overdamped, underdamped, or critically damped.





Overdamped

$$\alpha = \frac{R}{2L} = \frac{400}{2(.02)} = 10{,}000 \ Np/s$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{[(.02)(1\times10^{-6})]^{1/2}}$$

$$= 7071 \ Rad/s$$

$$\alpha > \omega_0 \longrightarrow \text{overdamped}$$

Critically damped

$$\alpha = \frac{R}{2L} = \frac{4}{2(2)} = 1 \ Np/s$$

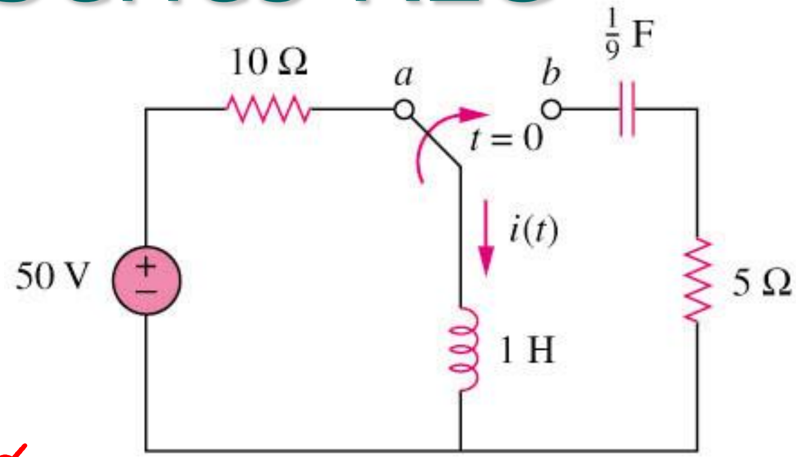
$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{[(2)(.5)]^{1/2}} = 1 \ Rad/s$$

$$\alpha = \omega_0 \longrightarrow$$

critically damped

2) Source-Free Series RLC

The circuit shown has reached steady state at t = 0-. The capacitor is uncharged. If the make-before-break switch moves to position b at t = 0, calculate i(t) for t > 0.



1) Find α, ω_0 + damping behavior

$$\alpha = \frac{R}{2L} = \frac{5}{2(1)} = \underline{\underline{2.5}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(1)\,1/9}} = \underline{\underline{3}}$$

$\omega_0 > \alpha$ $\therefore$ underdamped,

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{9 - 6.25} = \underline{\underline{1.658}}$$

2) Solve for Initial conditions a) $i_L(0^+)$ + b) $\frac{di_L(0^+)}{dt}$

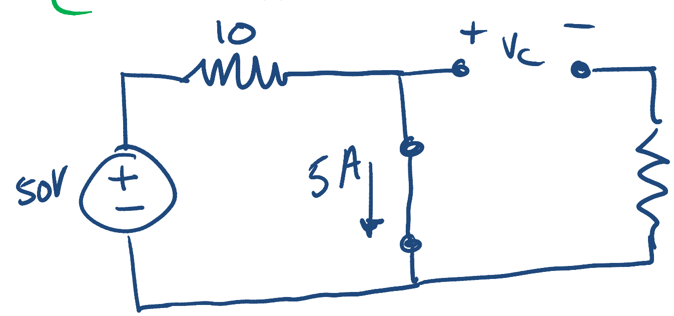
a) $i_L(0^+) \equiv i_L(0^-) = \frac{50V}{10\Omega} = 5A$ (Inductor is a short at $t=0^-$)

Inductor

Circuit at $t = 0^-$

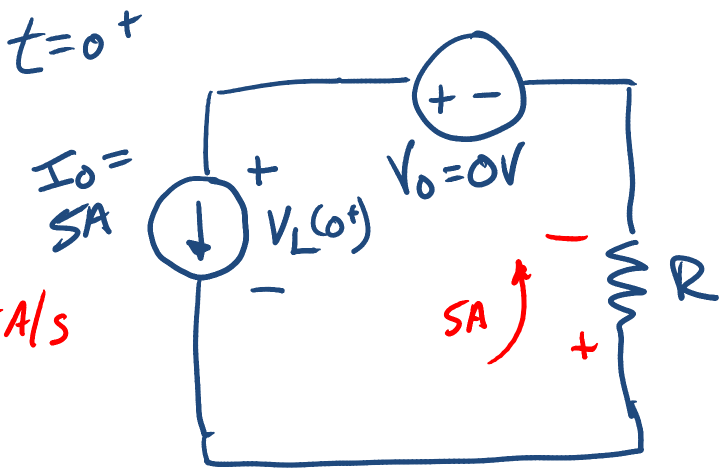
L → short cct

C → open cct



5) Inductor Law: $V_L = L\frac{di_L}{dt} \rightarrow \frac{di_L}{dt}(0^+) = \frac{V_L(0^+)}{L}$

Redraw ckt at $t=0^+$, with $L \rightarrow I_0$ & $C \rightarrow V_0$



KVL

$-V_L(0^+) + 0 - (5A)(5\Omega) = 0$

$V_L(0^+) = -25V$

$\frac{di_L}{dt}(0^+) = \frac{V_L(0^+)}{L} = \frac{-25}{1} = -25A/s$

3) Solve for B_1, B_2 (see table handout)

$B_1 = i_L(0^+) = 5$

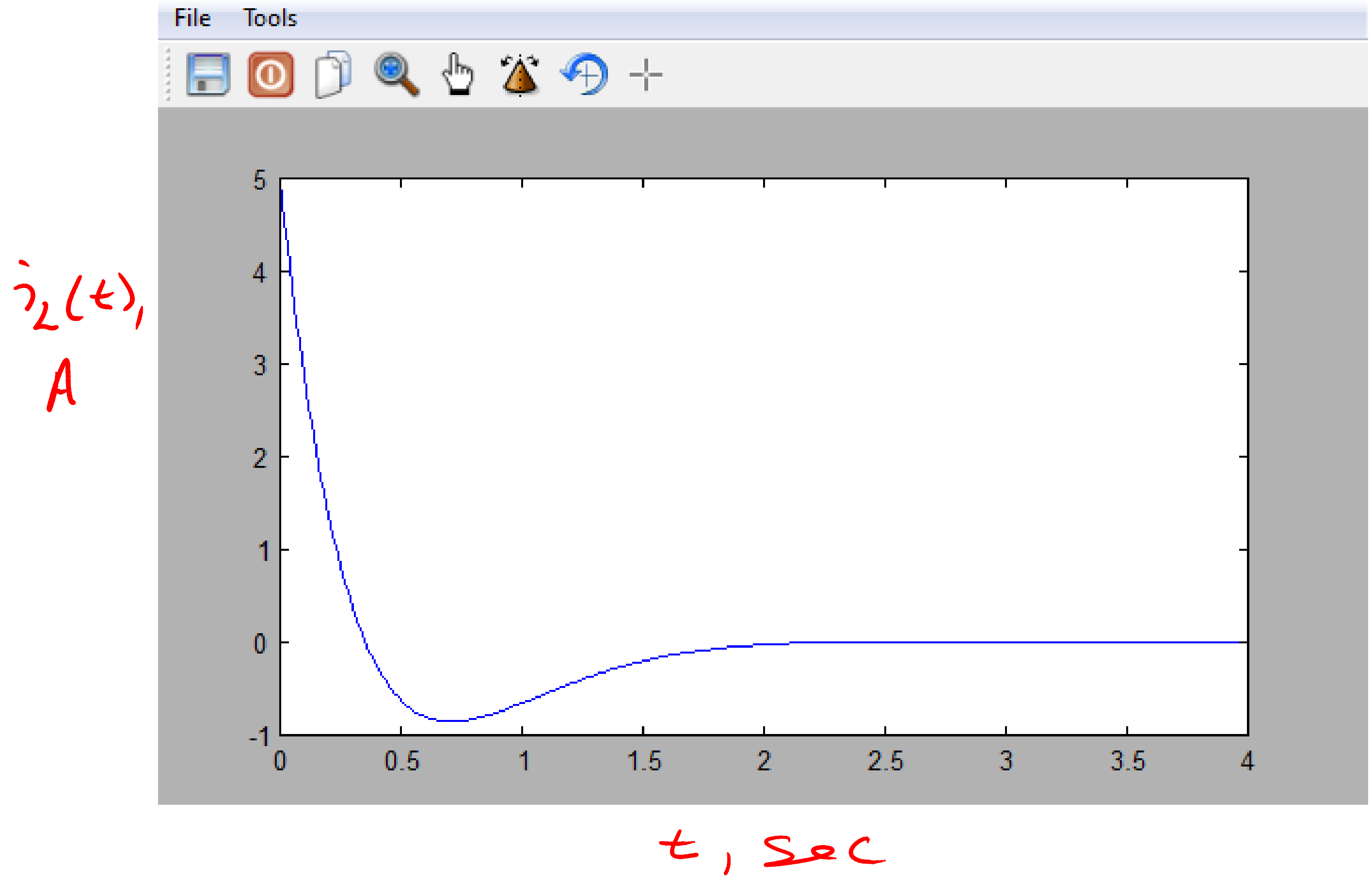
$-\alpha B_1 + \omega_d B_2 = \frac{di_L(0^+)}{dt} = -25 A/s$

$-(2.5)(5) + (1.658)B_2 = -25 \longrightarrow B_2 = \frac{-12.5}{1.658} = -7.539$

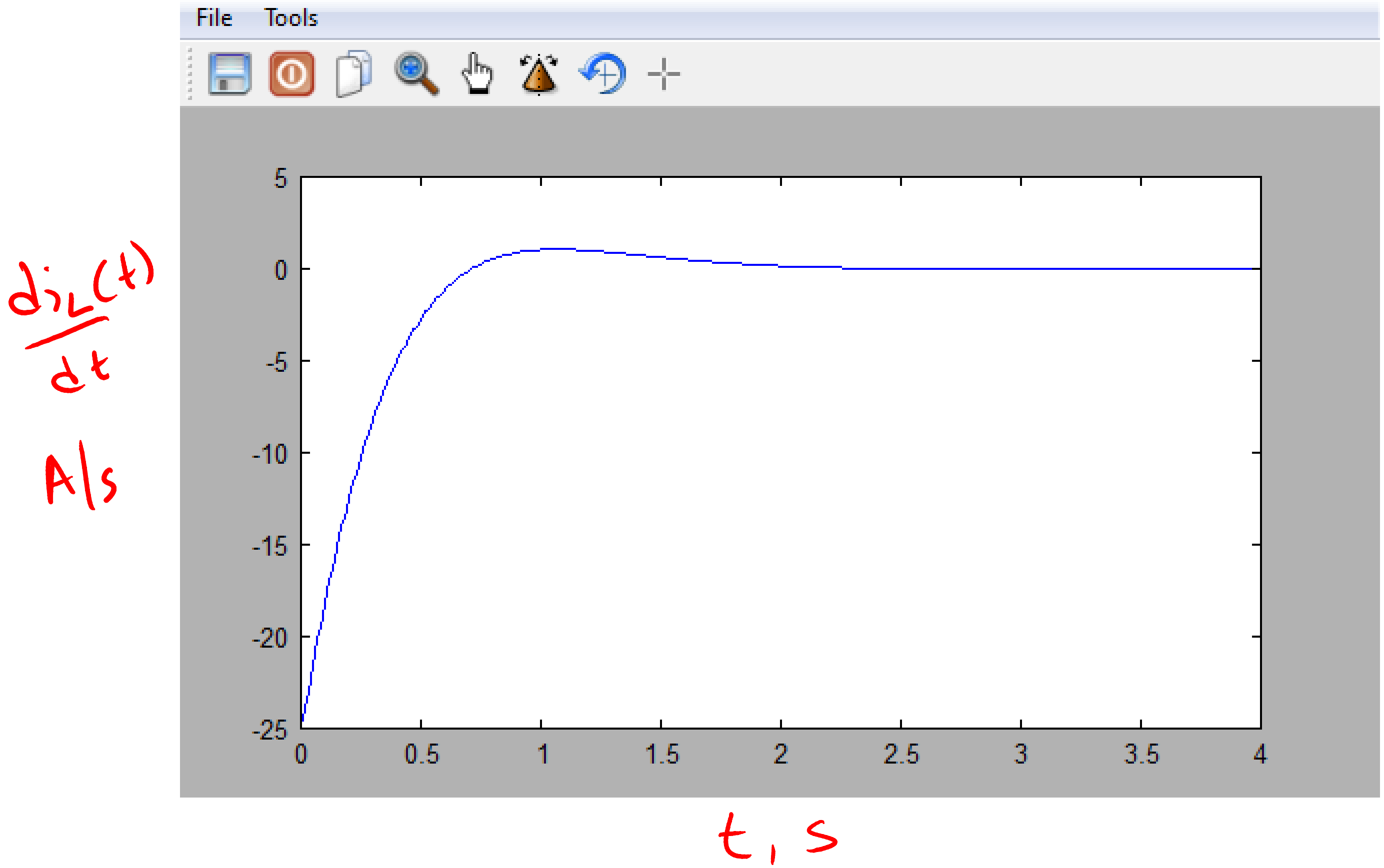
4) Compile Solution $i_L(t) = e^{-2.5t}(5\cos 1.658t - 7.539 \sin 1.658t)$

Answer: $i(t) = e^{-2.5t}[5\cos 1.6583t - 7.538\sin 1.6583t]$ A

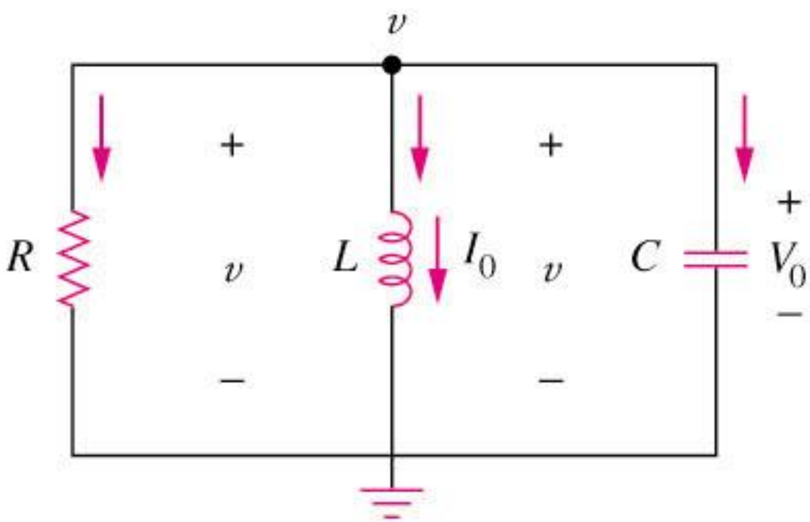
```
t = 0:0.01:4;
f=exp(-2.5*t).*(5*cos(1.6583*t)-7.538*sin(1.6583*t));
plot(t,f)
```



plot(t(2:end),diff(f)./diff(t))



8.3 Source-Free Parallel RLC Circuits



Let $$i(0) = I_0 = \frac{1}{L}\int_{-\infty}^{0} v(t)dt$$

$v(0) = V_0$

Apply KCL to the top node:

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v dt + C\frac{dv}{dt} = 0$$

Taking the derivative with respect to t and dividing by C

The 2nd order equation

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

8.3 Source-Free Parallel RLC Circuits

There are three possible solutions for the following 2nd order differential equation:

now using V_c as variable

$$\frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC} v = 0$$

=> $$\frac{d^2 v}{dt^2} + 2\alpha \frac{dv}{dt} + \omega_0^2 v = 0$$

General 2nd order Form

where

$$\alpha = \frac{1}{2RC} \quad and \quad \omega_0 = \sqrt{\frac{1}{LC}}$$

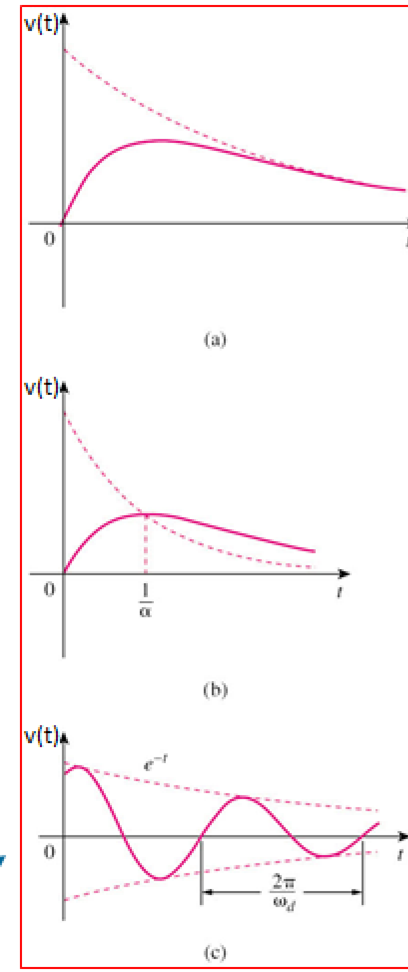
vs series RLC, where $\alpha = \frac{R}{2L}$

The types of solutions for i(t) depend on the relative values of α and ω.

8.3 Source-Free Parallel RLC Circuits

There are three possible solutions for the following 2nd order differential equation:

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega_0^2 v = 0$$



1. If $\alpha > \omega_o$, over-damped case

$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ **where** $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

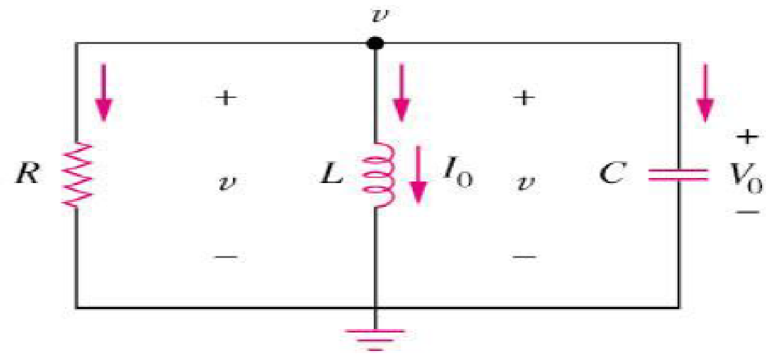
2. If $\alpha = \omega_o$, critical damped case

$v(t) = (A_2 + A_1 t)\, e^{-\alpha t}$ **where** $s_{1,2} = -\alpha$

3. If $\alpha < \omega_o$, under-damped case

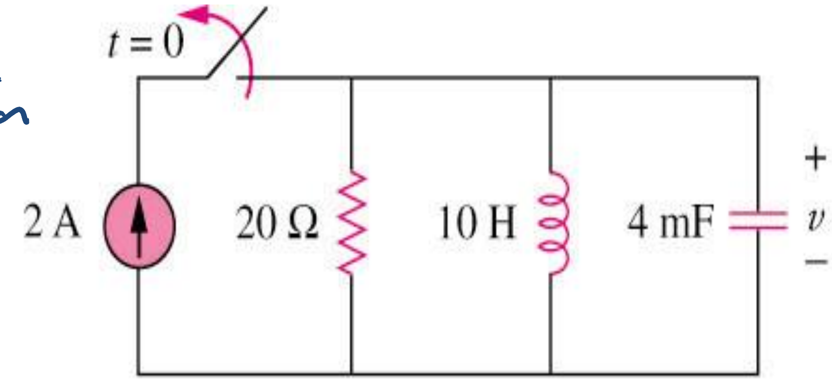
$v(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$ **where** $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

Table 3: Natural Response of Parallel RLC

Circuit			
Diff-Eq	$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega_0^2 v = 0$ where $\alpha = \frac{1}{2RC}$ and $\omega_0 = \sqrt{\frac{1}{LC}}$		
Damping	Over: $\omega_0^2 < \alpha^2$	Under: $\omega_0^2 > \alpha^2$	Critical: $\omega_0^2 = \alpha^2$
Form of Soln	$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$	$v(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$	$v(t) = (A_2 + A_1 t)\, e^{-\alpha t}$
Roots	$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$s_{1,2} = -\alpha \pm j\omega_d$, $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$	$s_{1,2} = -\alpha$
Boundary Conditions	$v(0+) = A_1 + A_2$ $\frac{dv(0+)}{dt} = \frac{i_C(0+)}{C} = s_1A_1 + s_2A_2$	$v_C(0+) = B_1$ $\frac{dv(0+)}{dt} = \frac{i_C(0+)}{C} = -\alpha B_1 + \omega_d B_2$	$v(0+) = A_2$ $\frac{dv(0+)}{dt} = \frac{i_C(0+)}{C} = A_1 - \alpha A_2$

3) Source-Free Parallel RLC Circuit

Find v(t) for t > 0.



1) Find α, ω_0 + damping behavior

$$\alpha = \frac{1}{2RC} = \frac{1}{2(20)(.004)} = 6.25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{[10(.004)]^{1/2}} = 5$$

$\alpha > \omega_0$ → overdamped →

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -6.25 + 3.75 = -2.5$$

$$s_2 = \ominus = -6.25 - 3.75 = -10$$

2) Solve for Initial Conditions a) $v(0^+)$ + b) $\frac{dv(0^+)}{dt}$

a) $v(0^+) \equiv v(0^-) = 0V$ (Inductor is a short at $t = 0^-$)

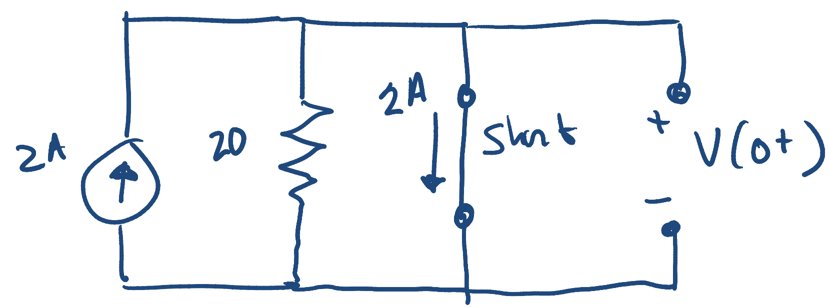
Capacitor

$i_L(0^+) = i_L(0^-) = 2\ A$

Redraw Cct at $t = 0^-$

L → short

C → open



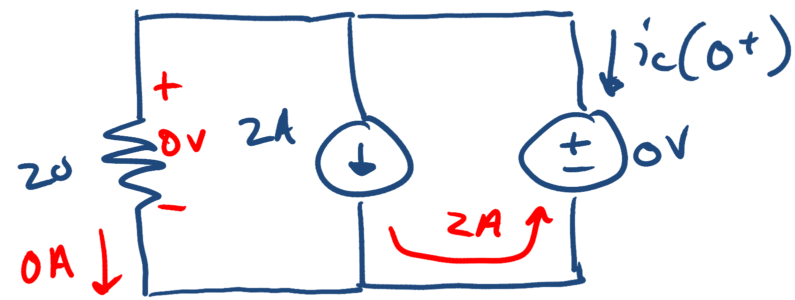
b) Capacitor Law: $i_c = C\frac{dV_c}{dt} \rightarrow \frac{dV_c(0^+)}{dt} = \frac{i_c(0^+)}{C}$

Redraw Cct at $t=0^+$, with $L \rightarrow I_0$ & $C \rightarrow V_0$

KCL

$i_c(0^+) = -2A$

$\frac{dV_c}{dt}(0^+) = \frac{-2A}{.004F} = -500 V/s$



3) Solve for A_1, A_2 (see table handout)

$$A_1 + A_2 = v(0^+) = 0 \rightarrow A_2 = -A_1$$

$$s_1A_1 + s_2A_2 = \frac{dv}{dt}(0^+) = -500 = -2.5A_1 - 10A_2$$

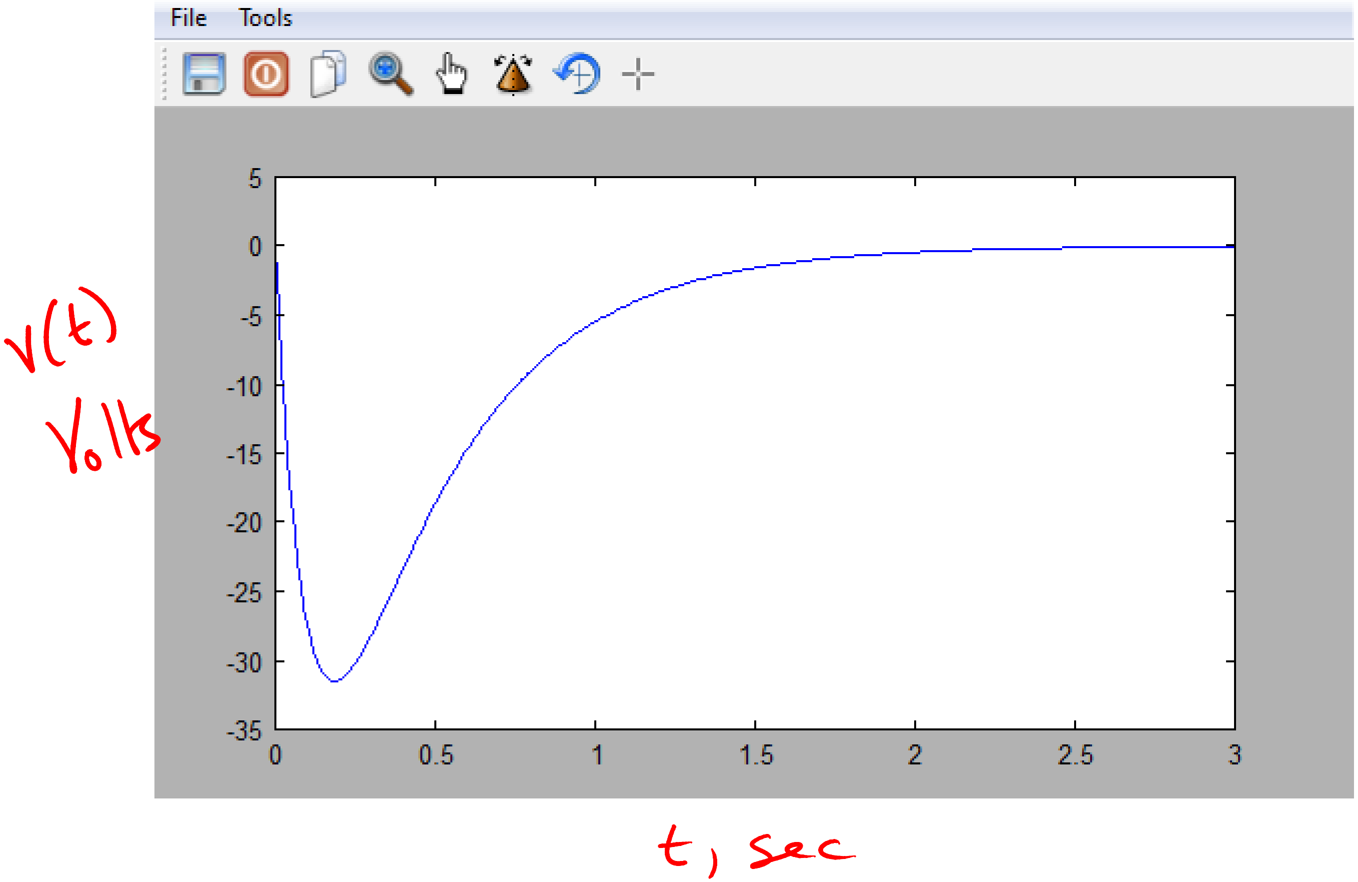
$$10A_1 - 2.5A_1 = -500,\quad 7.5A_1 = -500,\quad A_1 = -66.67,\ A_2 = +66.67$$

4) Assemble final solution:

$$v(t) = A_1e^{s_1t} + A_2e^{s_2t} = -66.67e^{-2.5t} + 66.67e^{-10t}$$

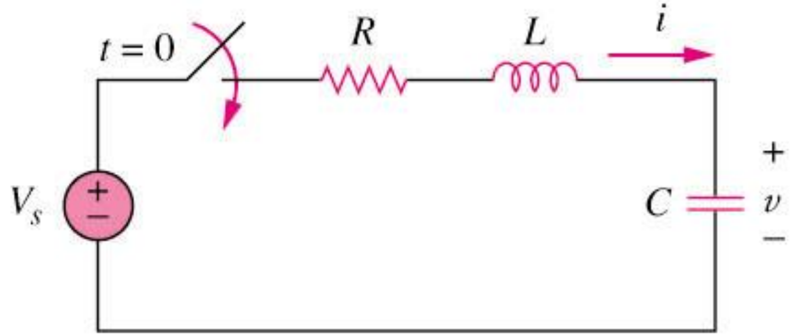
Answer: $v(t)$ = **$66.67(e^{-10t} - e^{-2.5t})$ V**

```
t = 0:0.01:4;
f = 66.67*(exp(-10*t)-exp(-2.5*t)) ;
plot(t,f)
axis([0 3 -35 5])
```



8.4 Step-Response Series RLC

- The step response is obtained by the sudden application of a dc source.



KVL $$Vs = Ri + L\frac{di}{dt} + v$$

But $$i = C\frac{dv}{dt}$$

Substitute, div by LC

The 2nd order expression

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{v_s}{LC}$$

The above equation has the same form as the equation for source-free series RLC circuit.

- The same coefficients (important in determining the frequency parameters).
- Different circuit variable in the equation.

8.4 Step-Response Series RLC Circuits (2)

The solution of the equation should have two components:
the transient response $v_t(t)$ & the steady-state response $v_{ss}(t)$:

$$v(t) = v_t(t) + v_{ss}(t)$$

- The transient response v_t is the same as that for source-free case

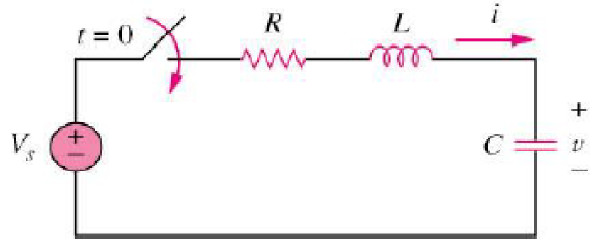
$$v_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(over-damped)}$$

$$v_t(t) = (A_1 + A_2 t) e^{-\alpha t} \quad \text{(critically damped)}$$

$$v_t(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) \quad \text{(under-damped)}$$

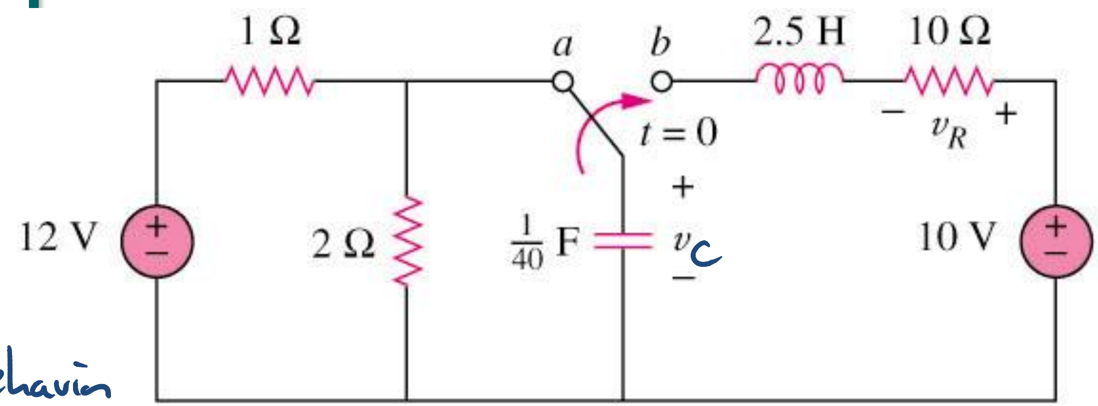
- The steady-state response is the final value of v(t)
  - $v_{ss}(t) = v(\infty)$
- The values of A_1 and A_2 are obtained from the initial conditions:
  - v(0) and dv(0)/dt.

Table 2: Step Response of Series RLC

Circuit			
Diff-Eq	$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{v_s}{LC}$	$\alpha = \frac{R}{2L}$ *and* $\omega_0 = \sqrt{\frac{1}{LC}}$	
Damping	Over: $\omega_0^2 < \alpha^2$	Under: $\omega_0^2 > \alpha^2$	Critical: $\omega_0^2 = \alpha^2$
Form of Soln	$v(t) = v(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$	$v(t) = v(\infty) + e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$	$v(t) = v(\infty) + A_1 t\, e^{-\alpha t} + A_2 e^{-\alpha t}$
Roots	$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$s_{1,2} = -\alpha \pm j\omega_d$, $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$	$s_{1,2} = -\alpha$
Boundary Conditions	$v(0+) = v(\infty) + A_1 + A_2$ $\frac{dv(0+)}{dt} = s_1 A_1 + s_2 A_2$	$v(0+) = v(\infty) + B_1$ $\frac{dv(0+)}{dt} = -\alpha B_1 + \omega_d B_2$	$v(0+) = v(\infty) + A_2$ $\frac{dv(0+)}{dt} = A_1 - \alpha A_2$

4) Step-Response Series RLC

Having been in position a for a long time, the switch in the circuit below is moved to position b at t = 0. Find v(t) and $v_R(t)$ for t > 0.



1) Find α, ω_0 + damping behavior

$$\alpha = \frac{R}{2L} = \frac{10}{(2)(2.5)} = 2$$

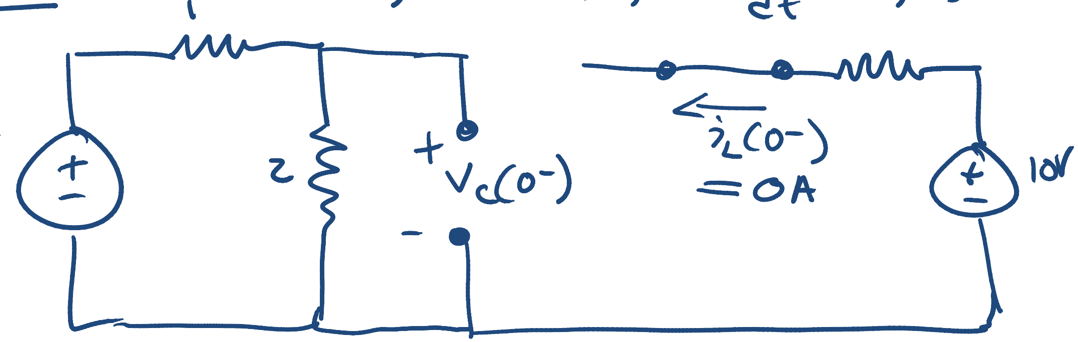
$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\left(\frac{2.5}{40}\right)^{\frac{1}{2}}} = 4$$

$\omega_0 > \alpha \rightarrow$ underdamped

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 3.46$$

2) Solve for initial + final conditions a) $v_c(0^+)$, b) $\frac{dv_c}{dt}(0^+)$, c) $v_c(\infty)$

Redraw cct at $t = 0^-$



a) $v_c(0^+) \equiv v_c(0^-)$

$$= 12\left(\frac{2}{2+1}\right) = 8V$$

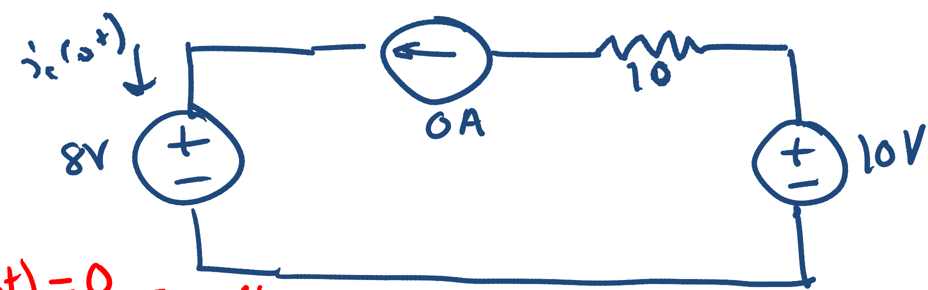
$$i_L(0^+) \equiv i_L(0^-) = 0A$$

b) $\frac{dv_c}{dt}(0^+) = \frac{i_c(0^+)}{C}$

$C \rightarrow 8V$

$L \rightarrow 0A$

(Redraw Cct at $t=0^+$)



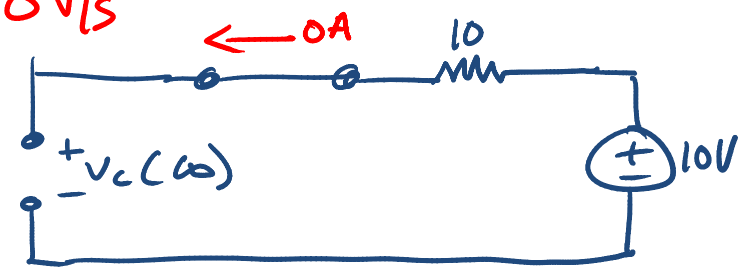
$i_c(0^+) = 0A$ (Inductor), $\frac{dv_c(0^+)}{dt} = \frac{0}{C} = 0\ V/s$

c) $v_c(\infty) = 10V$

$C \rightarrow$ open

$L \rightarrow$ short

(Redraw cct at $t = \infty$)



3) Solve for B_1, B_2

$v_c(0^+) = v_c(\infty) + B_1 \rightarrow B_1 = v(0^+) - v(\infty) = 8 - 10 = -2V$

$\frac{dv_c(0^+)}{dt} = 0 = -\alpha B_1 + \omega_d B_2 = -(2)(-2) + 3.46 B_2 = 0$

$B_2 = \frac{-4}{3.46} = -1.15$

4) Assemble final solution

$v_c(t) = v(\infty) + e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$

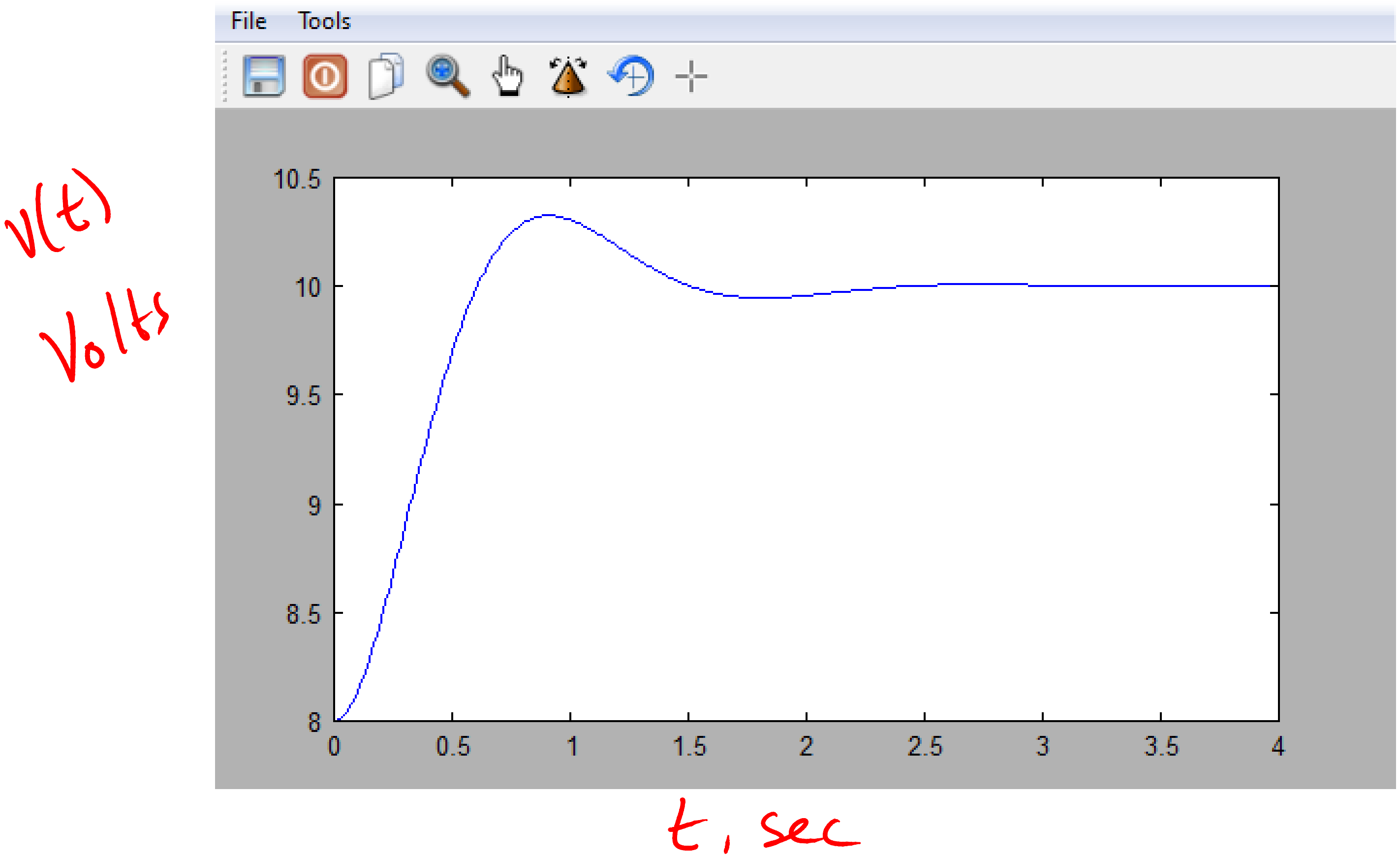
$= 10 + e^{-2t}(-2\cos(3.46t) - 1.15\sin(3.46t))$

$v_R(t) = R\, i_c(t) = R\, dv_c(t)/dt$

Answer: $v(t)$ = **{10 + [(−2cos3.464t − 1.1547sin3.464t)e−2t]} V**

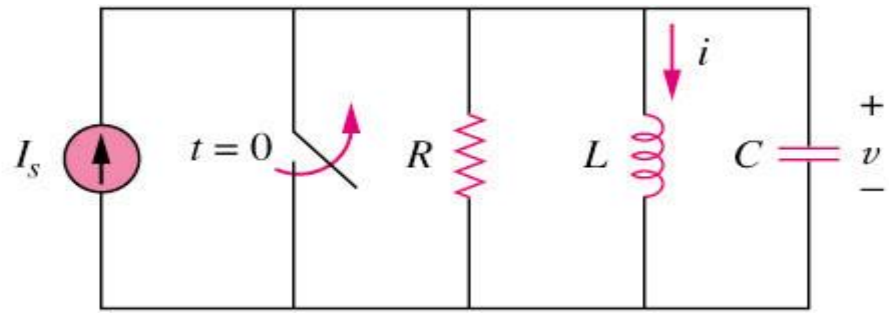
$v_R(t)$ = **[2.31sin3.464t]e^{-2t} V**

```
t = 0:0.01:4;
f = 10+(-2*cos(3.464*t)-1.1547*sin(3.464*t))*exp(-2*t);
plot(t,f)
```



8.5 **Step-Response** Parallel RLC Circuits (1)

- The step response is obtained by the sudden application of a dc source.



The 2nd order expression

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

It has the same form as the equation for source-free parallel RLC circuit.

- The same coefficients (important in determining the frequency parameters).
- Different circuit variable in the equation.

8.5 Step-Response Parallel RLC Circuits (2)

The solution of the equation should have two components: the transient response $i_t(t)$ & the steady-state response $i_{ss}(t)$:

$$i(t) = i_t(t) + i_{ss}(t)$$

- The transient response i_t is the same as that for source-free case

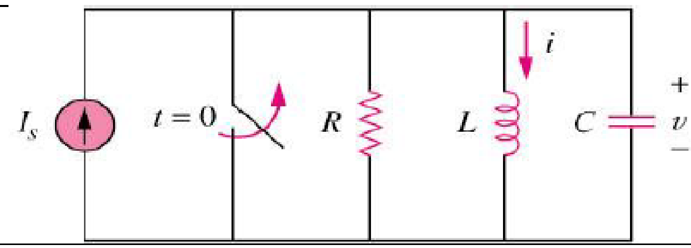
$$i_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(over-damped)}$$

$$i_t(t) = (A_1 + A_2 t) e^{-\alpha t} \qquad \text{(critical damped)}$$

$$i_t(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) \qquad \text{(under-damped)}$$

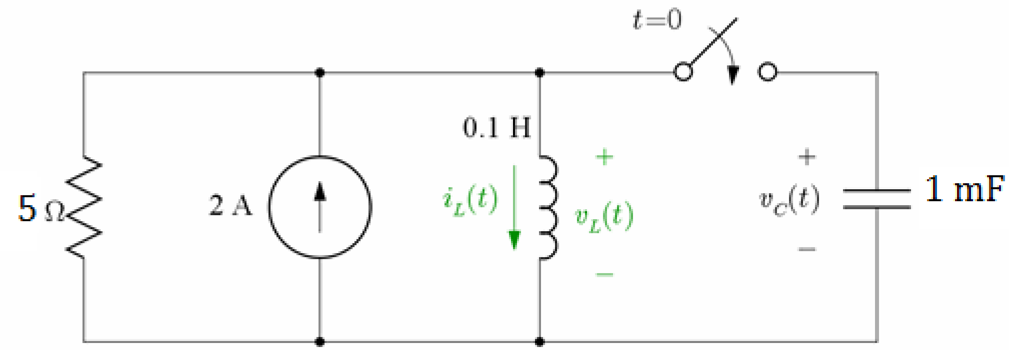
- The steady-state response is the final value of i(t).
  - $i_{ss}(t) = i(\infty) = I_s$
- The values of A_1 and A_2 are obtained from the initial conditions:
  - i(0) and di(0)/dt.

Table 4: Step Response of Parallel RLC

Circuit			
Diff-Eq	$\frac{d^2i}{dt^2}+\frac{1}{RC}\frac{di}{dt}+\frac{i}{LC}=\frac{I_s}{LC}$ where $\alpha=\frac{1}{2RC}$ and $\omega_0=\sqrt{\frac{1}{LC}}$		
Damping	Over: $\omega_0^2 < \alpha^2$	Under: $\omega_0^2 > \alpha^2$	Critical: $\omega_0^2 = \alpha^2$
Form of Soln	$i(t) = i(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$	$i(t) = i(\infty) +$ $e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$	$i(t) = i(\infty) + A_1 t\, e^{-\alpha t} + A_2 e^{-\alpha t}$
Roots	$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$s_{1,2} = -\alpha \pm j\omega_d$, $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$	$s_{1,2} = -\alpha$
Boundary Conditions	$i(0+) = i(\infty) + A_1 + A_2$ $\frac{di(0+)}{dt} = \frac{v_L(0+)}{L} = s_1A_1 + s_2A_2$	$i(0+) = i(\infty) + B_1$ $\frac{di(0+)}{dt} = \frac{v_L(0+)}{L} = -\alpha B_1 + \omega_d B_2$	$i(0+) = i(\infty) + A_2$ $\frac{di(0+)}{dt} = \frac{v_L(0+)}{L} = A_1 - \alpha A_2$

5)Step-Response Parallel RLC Cct

Find $i_L(t)$ for t>0, given $v_c(0-)$=10V, and switch has been open a long time before closing at t=0



1) find α, ω_o & damping

$$\alpha = \frac{1}{2RC} = \frac{1}{2(5)(.001)} = 100$$

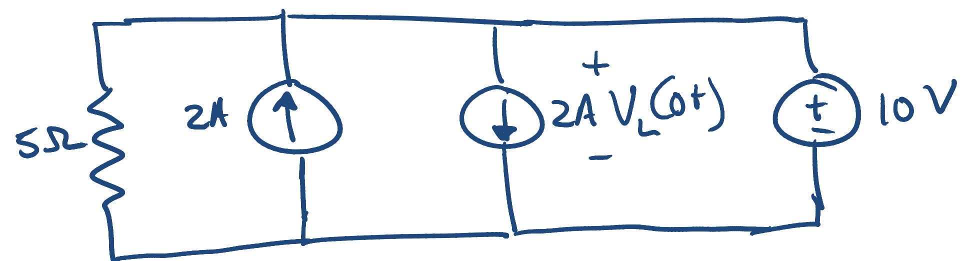
$$\omega_o = \frac{1}{\sqrt{(.1)(.001)}} = 100$$

$\omega_o = \alpha \rightarrow$ Critically Damped

2) Solve for Initial & final conditions a) $i_L(0^+)$, b) $\frac{di_L(0^+)}{dt}$, c) $i_L(\infty)$

a) $i_L(0^+) = 2A$

Redraw at $t=0^+$ $C \rightarrow V_o$, $L \rightarrow I_o$



b) $\frac{di_L(0^+)}{dt} = \frac{V_L(0^+)}{L}$

$V_L(0^+) = V_C(0^+) = 10V$

$$\frac{di_L(0^+)}{dt} = \frac{10V}{0.1H} = 100 \text{ V/s}$$

c) $i_L(\infty) = 2A$

3) Solve for A_1 & A_2 (tasks)

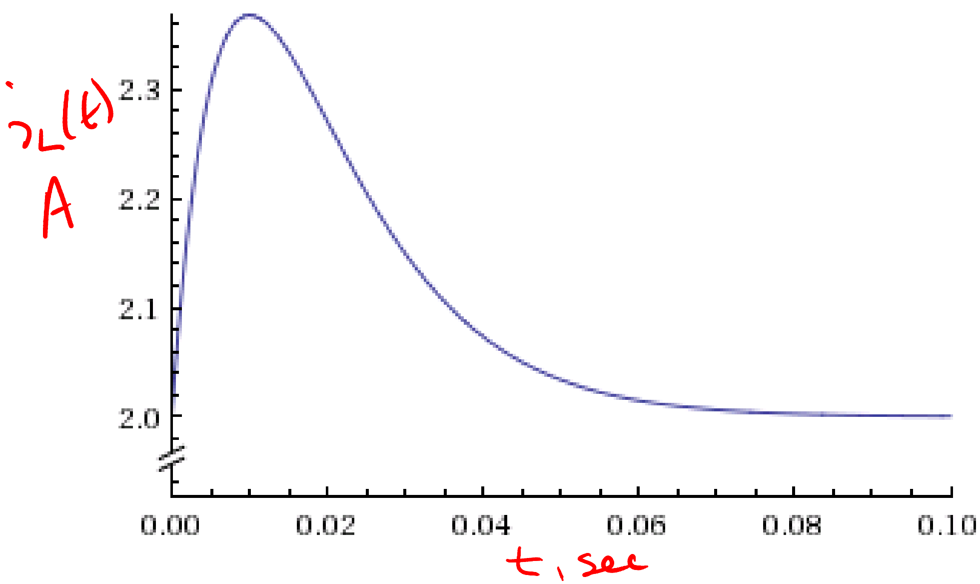
$$i_L(0^+) = A_2 + i_L(\infty), \quad A_2 = i_L(0^+) - i_L(\infty) = 0$$

$$\frac{di_L}{dt}(0^+) = A_1 - \alpha A_2 = 100, \quad A_1 = 100$$

4) Assemble final solution

$$i_L(t) = i_L(\infty) + A_1 t e^{-\alpha t} + A_2 e^{-\alpha t}$$

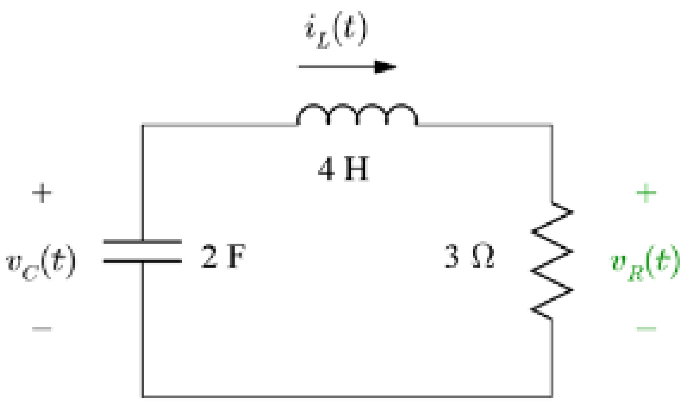
$$= 2 + 100 t e^{-100t}$$



Answer: $i_L(t) = 2 + 100te^{-100t}, \quad t>0$

6) Determine the initial value of Vr(t), and dVr(t)/dt, and the final value of Vr(t)

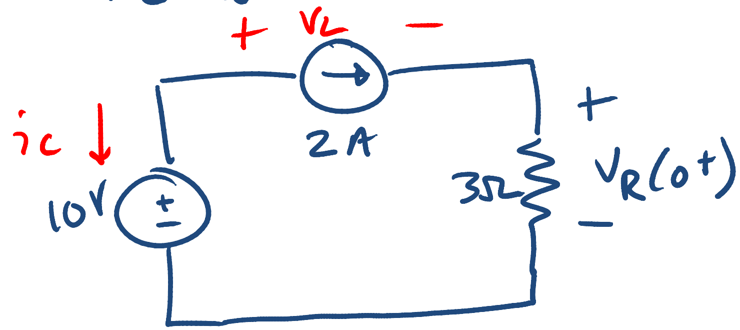
http://www.rose-hulman.edu/CLEO/video/player.php?id=34&embed



$v_C(0) = 10\ \text{V}$

$i_L(0) = 2\ \text{A}$

Redraw Circuit at $t = 0^+$



Ohm's Law

$$V_R(0^+) = i_R(0^+)\,3 = 6V$$

$$\frac{dv_C}{dt}(0^+) = \frac{i_C}{C} = -\frac{2}{2} = -1\,V/s$$

$$\frac{d i_L}{dt}(0^+) = \frac{V_L}{L} = \frac{4}{4} = 1\,A/s$$

$$i_L(\infty) = 0 \quad (\text{capacitor})$$

$$\frac{dV_R}{dt}(0^+) = \frac{d i_R}{dt}(0^+)\,3$$

$$= \frac{d i_L}{dt}(0^+)(3)$$

$$= (1)(3) = 3\,A/s$$

$$V_R(\infty) = i_R(\infty)(3)$$

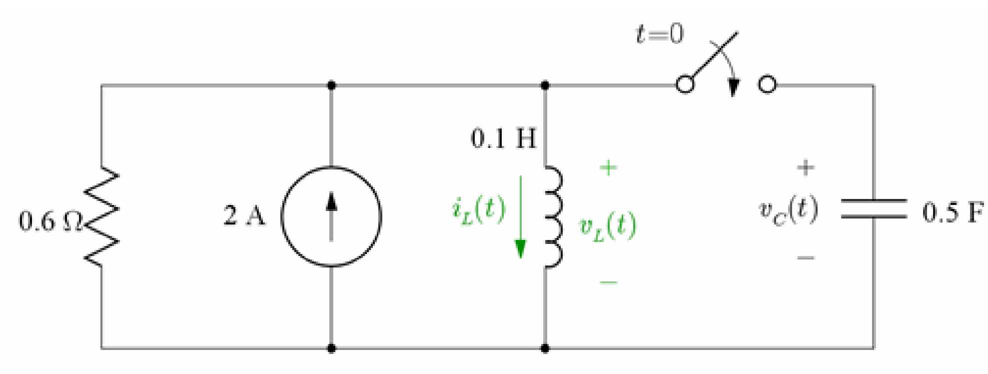
$$= i_L(\infty)(3) = \underline{0}$$

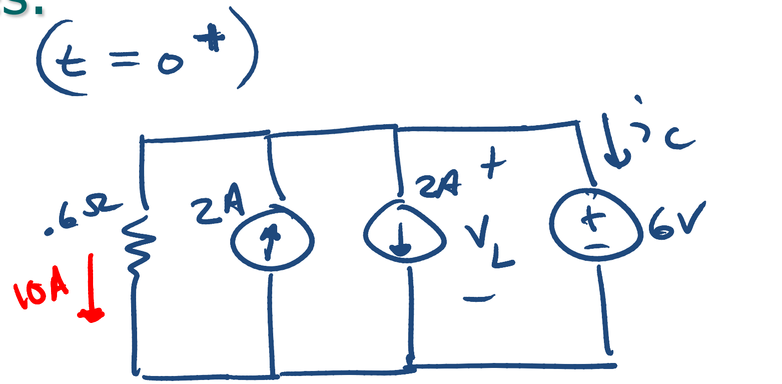
Vr(0) = 6V;

dVr(0)/dt = 3V/s;

Vr(infinity) = 0V

7) For both the inductor current and voltage, determine their initial values, the initial values of their derivatives, and their final values. The capacitor has 9 J of stored energy before the switch closes.





$W_c = 9\,J = \frac{CV_c^2}{2} = \frac{.5\,V_c^2}{2} \rightarrow V_c^2 = 9(4),\ \underline{V_c(0^+) = 6V}$

$i_L(0^+) = \underline{2A}$

$\frac{di_L}{dt}(0^+) = \frac{V_L(0^+)}{L} = \frac{V_c(0^+)}{L} = \frac{6}{.1} = \underline{60\ A/s}$

$i_L(\infty) = \underline{2A}$

$\frac{dV_c}{dt}(0^+) = \frac{i_c(0^+)}{C} = \frac{-10A}{.5F}$

$= \underline{-20\ V/s}$

$V_c(\infty) = \underline{0V}$

$i_L(0) = 2A$ $\quad$ $v_L(0^-) = 0V,\ v_L(0^+) = 6V$

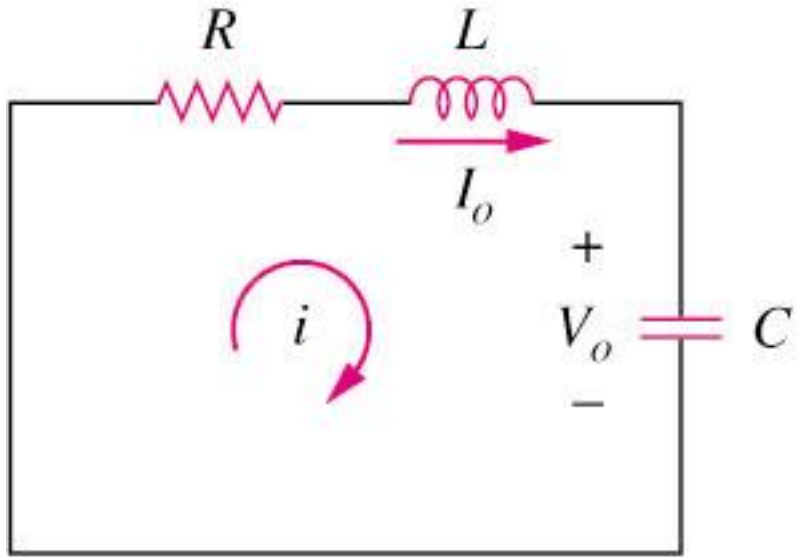
$\frac{di_L(0)}{dt} = 60\ A/s$ $\quad$ $\frac{dv_L(0)}{dt} = -20\ V/s$

$i_L(\infty) = 2A$ $\quad$ $v_L(\infty) = 0V$

Handouts

1a) Source-Free Series RLC

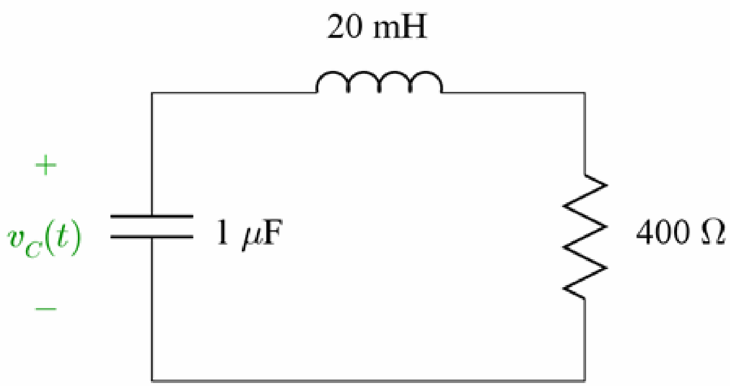
If R = 10 Ω, L = 5 H, and C = 2 mF, find α, ω_o, s1 and s2.



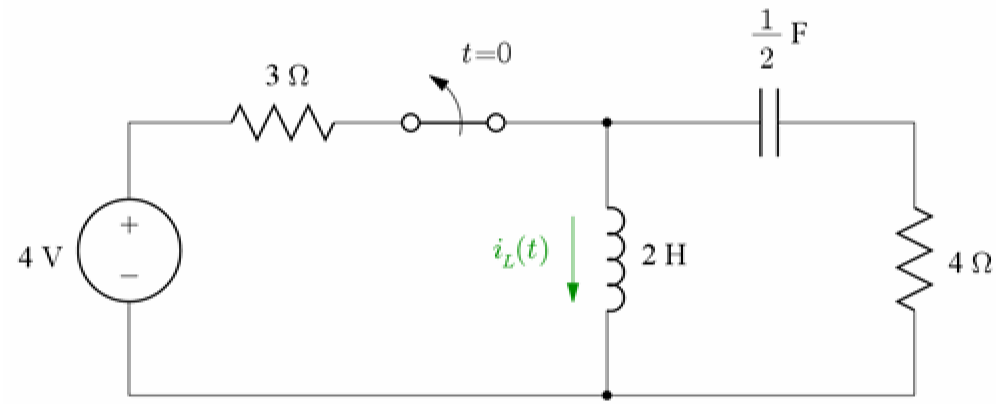
What type of natural response will the circuit have?

Answer: *underdamped*

1b)determine the qualitative form of the response as being either overdamped, underdamped, or critically damped.



Overdamped

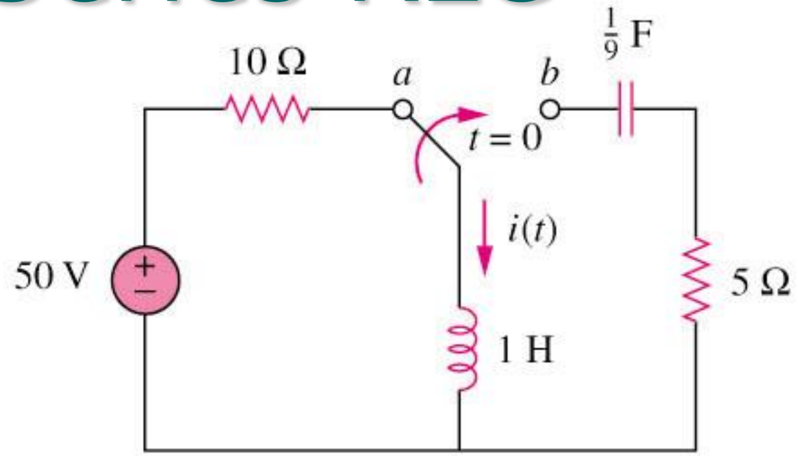


Critically damped

2) Source-Free Series RLC

The circuit shown has reached steady state at t = 0-.

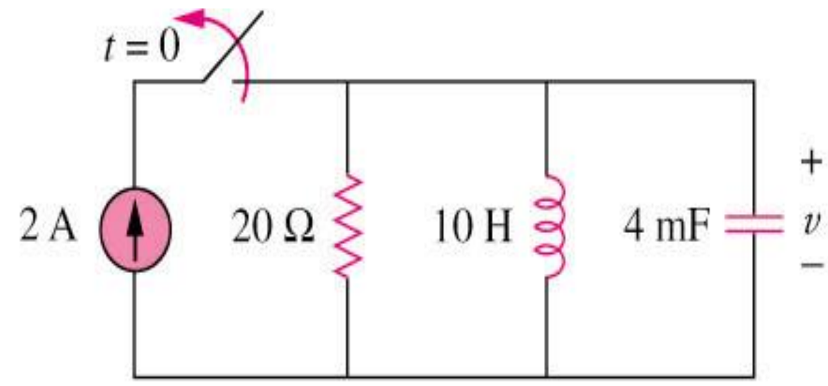
If the make-before-break switch moves to position b at t = 0, calculate i(t) for t > 0.



Answer: $i(t) = e^{-2.5t}[5\cos 1.6583t - 7.538\sin 1.6583t]$ A

3) Source-Free Parallel RLC Circuit

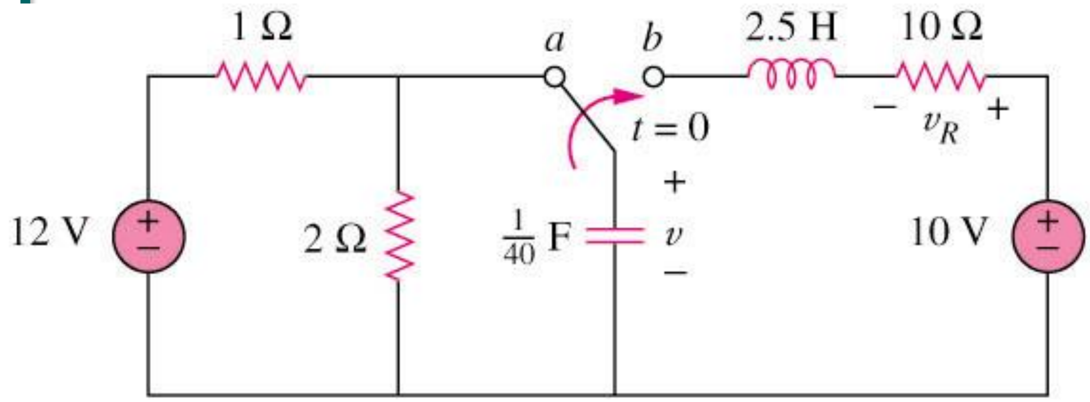
Find v(t) for t > 0.



Answer: $v(t)$ = **$66.67(e^{-10t} - e^{-2.5t})$ V**

4) Step-Response Series RLC

Having been in position a for a long time, the switch in the circuit below is moved to position b at t = 0. Find v(t) and $v_R(t)$ for t > 0.

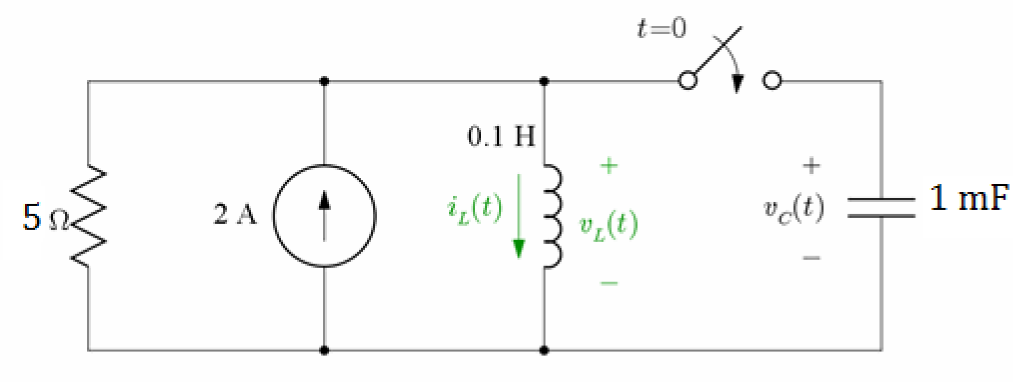


Answer: $v(t)$ = ***{10 + [(−2cos3.464t − 1.1547sin3.464t)e−2t]} V***

$v_R(t)$ = ***[2.31sin3.464t]*** e^{-2t} ***V***

5)Step-Response Parallel RLC Cct

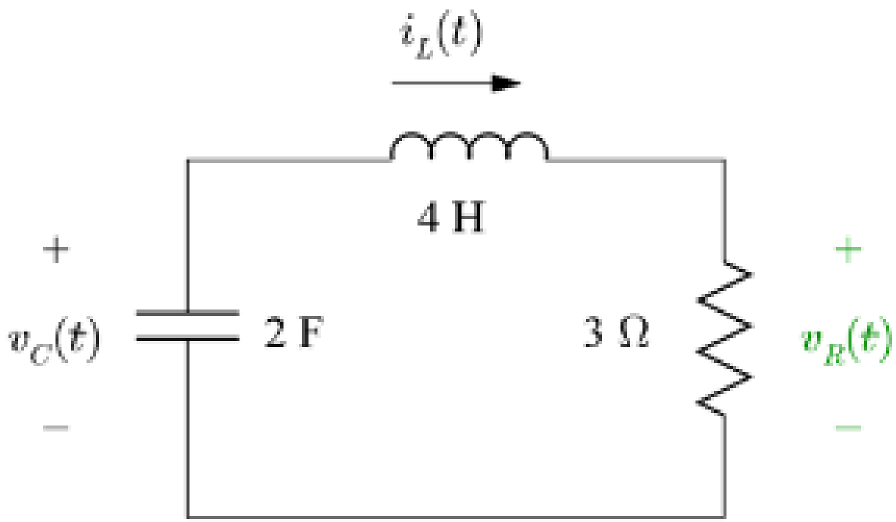
Find $i_L(t)$ for t>0, given $v_c(0-)$=10V, and switch has been open a long time before closing at t=0



Answer: ***$i_L(t) = 2 + 100te^{-100t}$, t>0***

6) Determine the initial value of Vr(t), and dVr(t)/dt, and the final value of Vr(t)

http://www.rose-hulman.edu/CLEO/video/player.php?id=34&embed

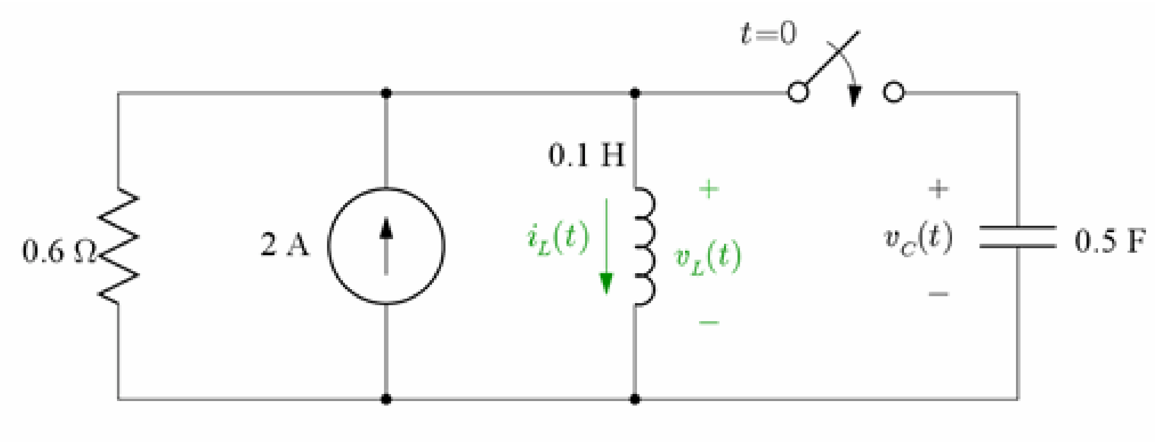


$v_C(0) = 10\,\text{V}$

$i_L(0) = 2\,\text{A}$

Vr(0) = 6V;
dVr(0)/dt = 3V/s;
Vr(infinity) = 0V

7) For both the inductor current and voltage, determine their initial values, the initial values of their derivatives, and their final values. The capacitor has 9 J of stored energy before the switch closes.



$i_L(0) = 2\,A$ $\quad v_L(0^-) = 0\,V,\ v_L(0^+) = 6\,V$

$\frac{di_L(0)}{dt} = 60\ A/s$ $\quad \frac{dv_L(0)}{dt} = -20\ V/s$

$i_L(\infty) = 2\,A$ $\quad v_L(\infty) = 0\,V$