Circuit Theory

Chapter 7
First-Order Circuits

see "Derivation" link for more information

First-Order Circuits
Chapter 7

"Natural Response"

7.1 The Source-Free RC Circuit

7.2 The Source-Free RL Circuit

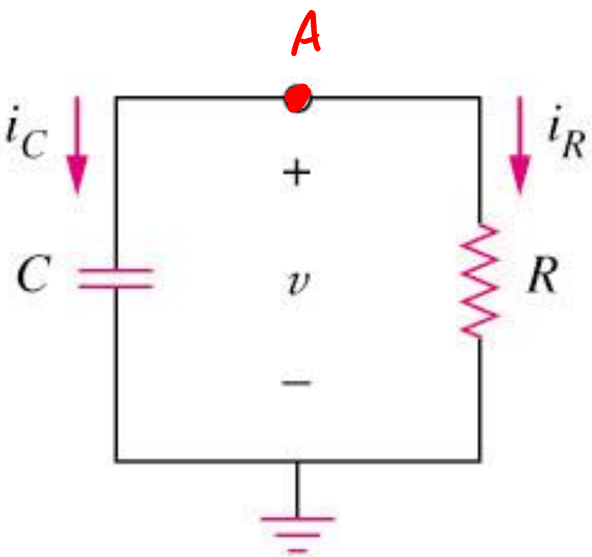
7.3 Unit-step Function

7.4 Step Response of an RC Circuit

7.5 Step Response of an RL Circuit

7.1 The Source-Free RC Circuit (1)

- A **first-order circuit** is characterized by a first-order differential equation.



KCL at Node A

$i_c + i_R = 0$

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

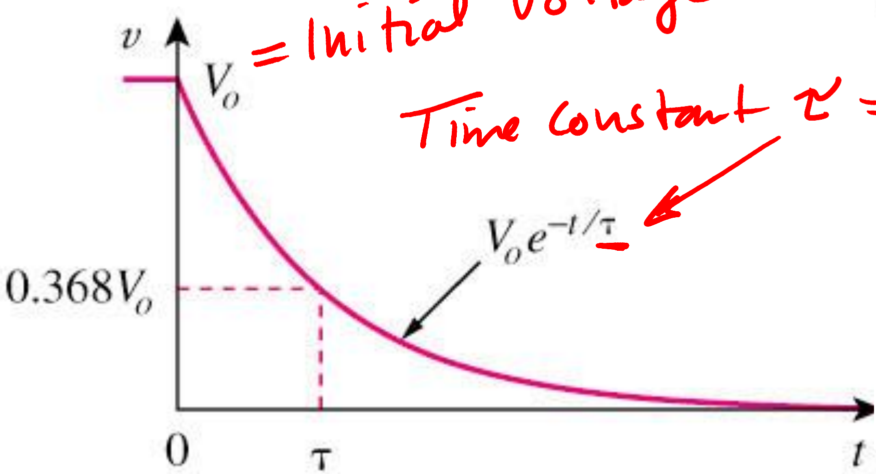
Capacitor Law

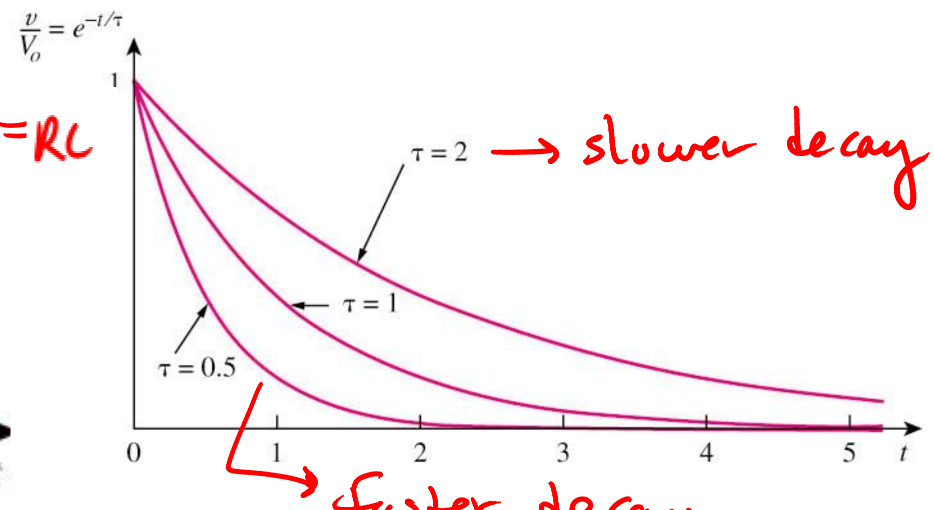
Ohm's Law

- Apply Kirchhoff's laws to <u>purely resistive circuit</u> results in <u>algebraic equations</u>.
- Apply the laws to RC and RL circuits produces <u>differential equations</u>.

7.1 The Source-Free RC Circuit (2)

- The **natural response** of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with <u>no external sources of excitation</u>.





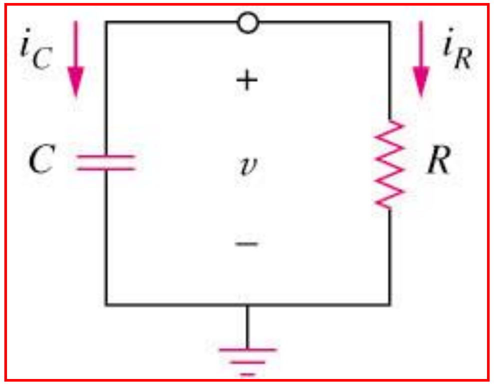
- The **<u>time constant</u>** τ of a circuit is the time required for the response to decay by a factor of <u>1/*e* or 36.8%</u> of its initial value.
- v decays faster for small τ and slower for large τ.

7.1 The Source-Free RC Circuit (3)

The key to working with a source-free RC circuit is finding:

a) V_0 b) τ

$$v(t) = V_0\, e^{-t/\tau}$$ where $$\tau = RC$$



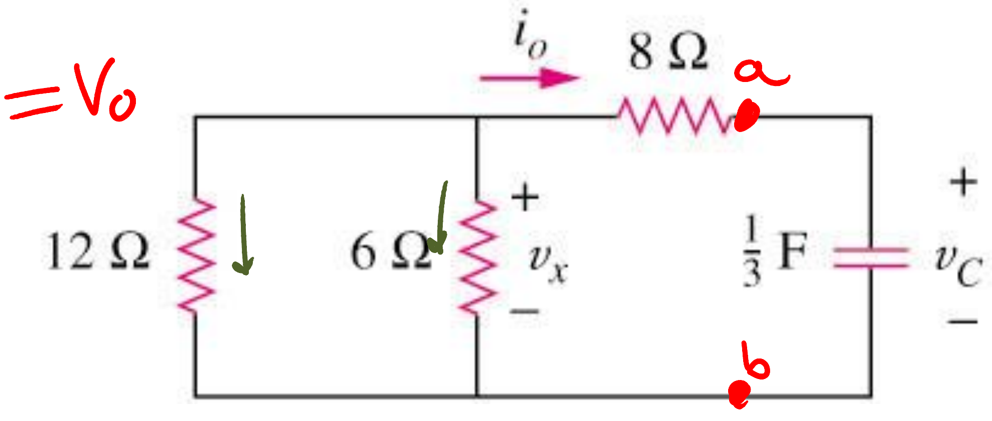
1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant τ = RC.

7.1 Source-Free RC Circuit (4)

Example 1

Refer to the circuit below, determine v_C, v_x, and i_o for $t \geq 0$.

Assume that $v_C(0) = 30$ V. $= V_0$



$\tau = R_{ab}C = (8 + 12 /\!/ 6)(\frac{1}{3})$

$= (8+4)(\frac{1}{3}) = 4$ seconds

$\therefore v_c(t) = V_0 e^{-t/\tau} = \underline{\underline{30 e^{-0.25t}}}$

V_x divides V_c

$V_x = \left(\frac{12/\!/6}{12/\!/6 + 8}\right) V_c = \frac{4}{12} V_c = \frac{1}{3} V_c = 10 e^{-.25t}$

KCL $i_0 + \frac{V_x}{12} + \frac{V_x}{6} = 0$, $i_0 = \frac{-V_x}{4} = -2.5e^{-.25t}$

OR $i_0 = i_c = C\frac{dV_c}{dt}$

$i_0 = (\frac{1}{3})(-\frac{1}{4}) 30 e^{-.25t}$

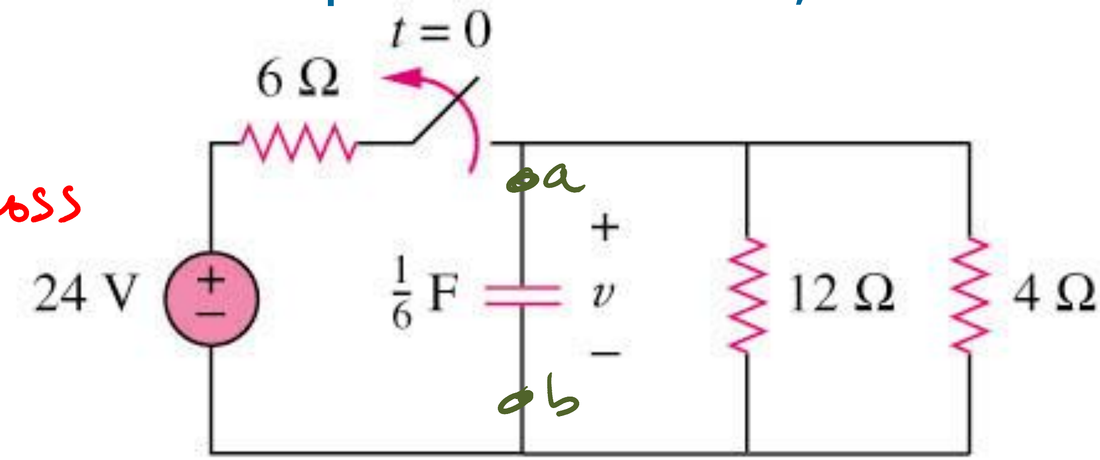
SAME

Answer: $v_C = 30e^{-0.25t}$ V ; $v_x = 10e^{-0.25t}$; $i_o = -2.5e^{-0.25t}$ A

7.1 Source-Free RC Circuit (5)

Example 2

The switch in circuit below is opened at t = 0, find v(t) for t ≥ 0.



PLAN

A) Find V_0, Voltage across parallel resistors (Voltage divider)

$$V_0 = 24\left(\frac{12/\!/4}{6+12/\!/4}\right) = \frac{24(3)}{6+3} = \underline{\underline{8\,V.}}$$

B) Find τ for $t \geq 0$, Rab seen by capacitor (switch open)

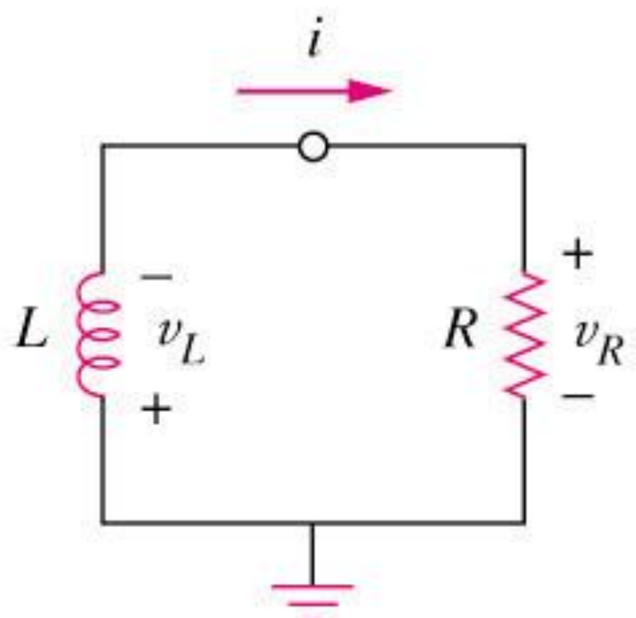
$$R_{ab} = 12/\!/4 = 3\,\Omega \quad , C = \tfrac{1}{6}F \rightarrow \tau = (3)\left(\tfrac{1}{6}\right) = 0.5$$

C) $V(t) = V_0 e^{-t/\tau} = 8e^{-t/\tau} = 8e^{-t/0.5} = \underline{\underline{8e^{-2t}}}$

Answer: $V(t) = 8e^{-2t}$ V

7.2 The Source-Free RL Circuit (1)

- A **first-order RL circuit** consists of a inductor L (or its equivalent) and a resistor (or its equivalent)



<u>KVL</u> $V_L + V_R = 0$

$$L\frac{di}{dt} + iR = 0$$

Inductor Law

Ohm's Law

time constant

<u>Solving</u> $\frac{di}{i} = -\frac{R}{L}dt \rightarrow i(t) = I_0 e^{-t/\tau}$

Initial Current

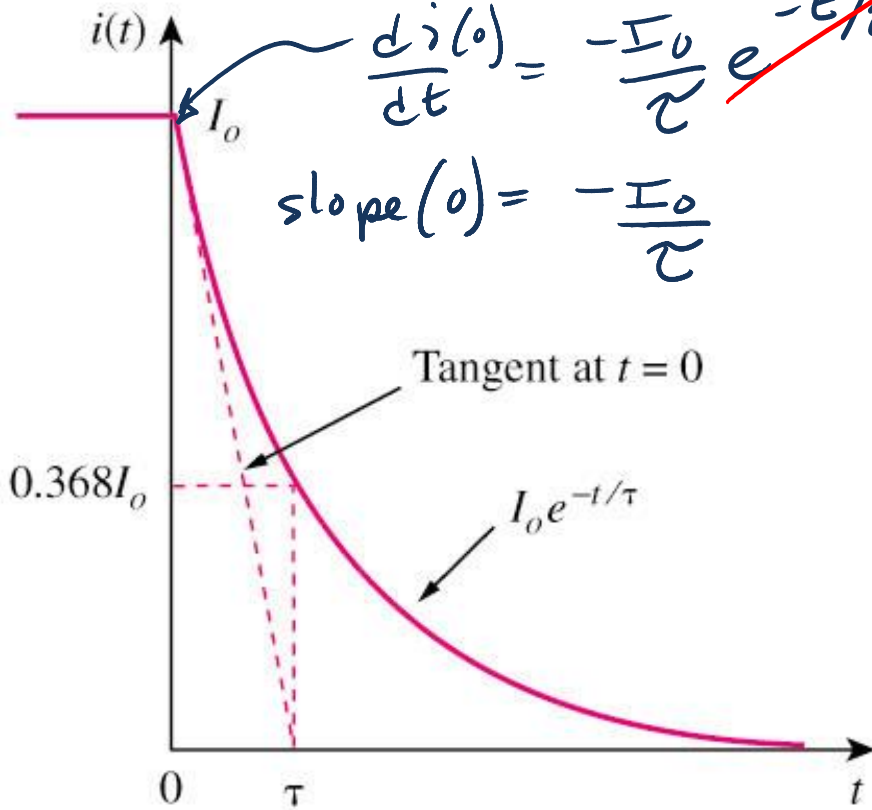
7.2 The Source-Free RL Circuit (2)

slope = $\frac{di(0)}{dt} = -\frac{I_0}{\tau} \cancel{e^{-t/\tau}} \rightarrow 1$

slope(0) = $-\frac{I_0}{\tau}$

$i(t) = I_0 e^{-t/\tau}$

where $\tau = \frac{L}{R}$



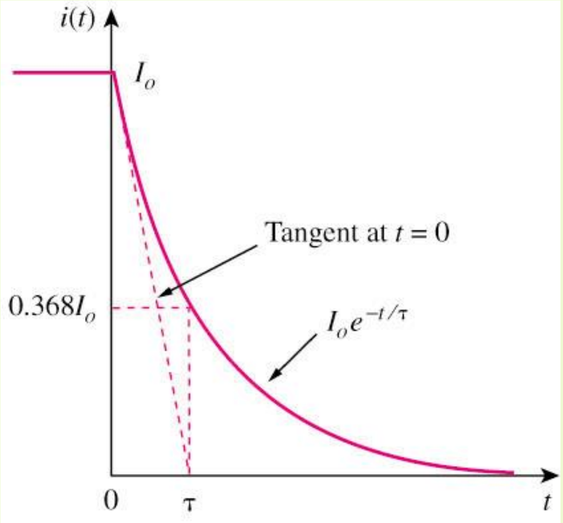
- The **time constant** τ of a circuit is the time required for the response to decay by a factor of 1/*e* or 36.8% of its initial value.
- *i(t)* decays faster for small τ and slower for large τ.
- The general form is very similar to a RC source-free circuit.

7.2 The Source-Free RL Circuit (3)

Comparison between a RL and RC circuit

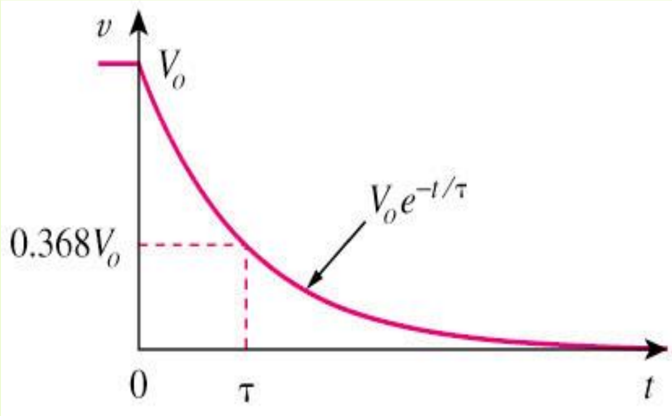
A RL source-free circuit

$i(t) = I_0\, e^{-t/\tau}$ **where** $\tau = \dfrac{L}{R}$



A RC source-free circuit

$v(t) = V_0\, e^{-t/\tau}$ **where** $\tau = RC$



7.2 The Source-Free RL Circuit (4)

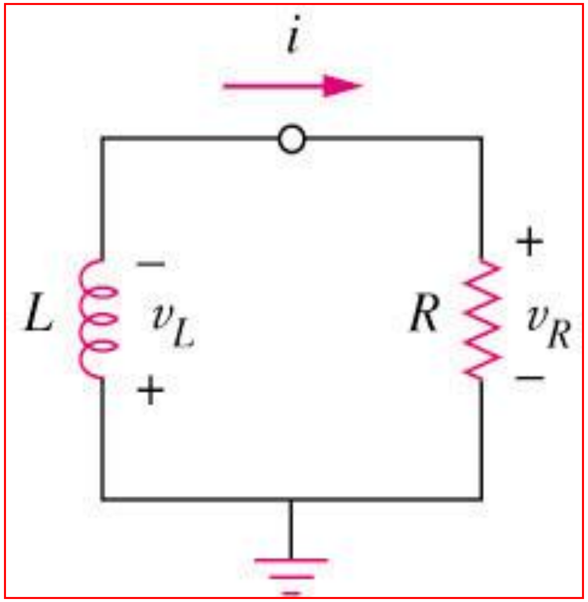
The key to working with a source-free RL circuit is finding:

a) I_0, initial current b) τ

$$i(t) = I_0 \, e^{-t/\tau}$$

where

$$\tau = \frac{L}{R}$$



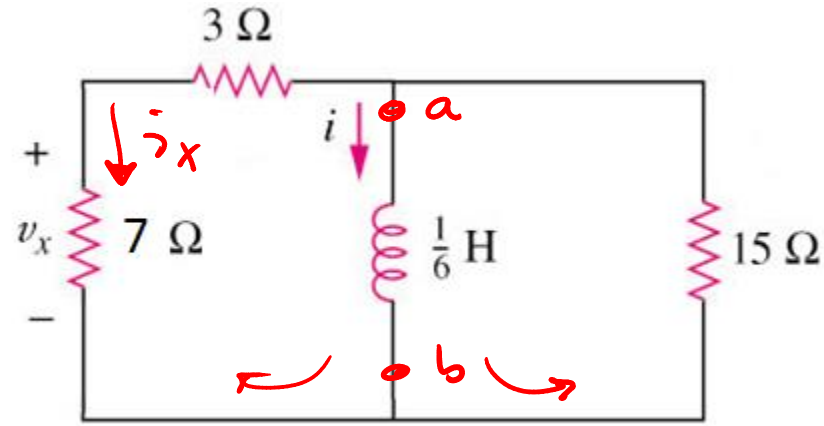
1. The initial voltage $i(0) = I_0$ through the inductor.
2. The time constant τ = L/R.

7.2 The Source-Free RL Circuit (5)

Example 3

Find i and v_x in the circuit.

Assume that $i(0) = 5$ A.



$i(t) = I_0 e^{-t/\tau} = 5e^{-t/\tau}$

$\tau = \frac{L}{R}$

$R = R_{ab}$ with Inductor Removed

$= 15 \,//\, (7+3) = 6\Omega$

$\tau = \frac{\frac{1}{6}}{6} = \frac{1}{36}$ sec

$\therefore i(t) = 5e^{-36t}$ A

Current Divider

$i_x = -i\left(\frac{15}{10+15}\right)$

$= -3e^{-36t}$

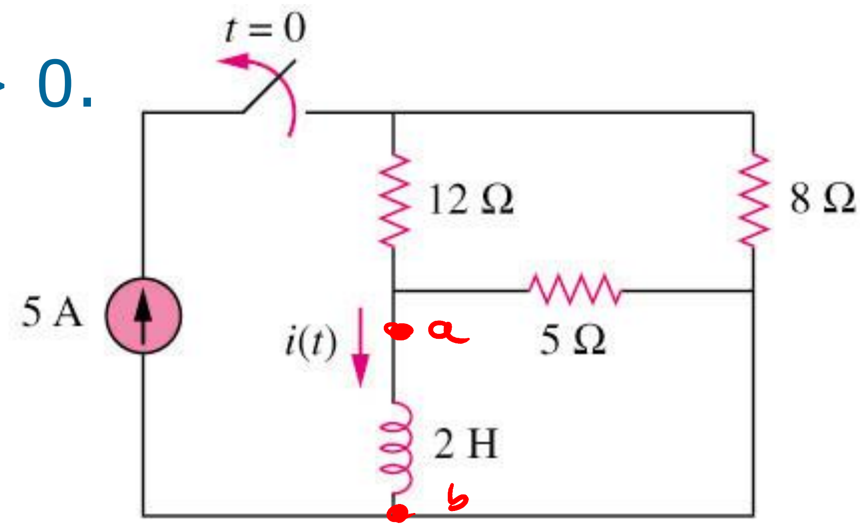
$V_x = 7 i_x = -21 e^{-36t}$

Answer: $i(t) = 5e^{-36t}$ A

7.2 The Source-Free RL Circuit (6)

Example 4

For the circuit, find i(t) for t > 0.

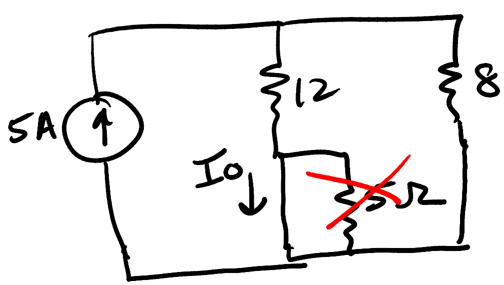


A) $t < 0$ → switch closed

Assume L is a short

Solve for I_o

(redraw circuit)

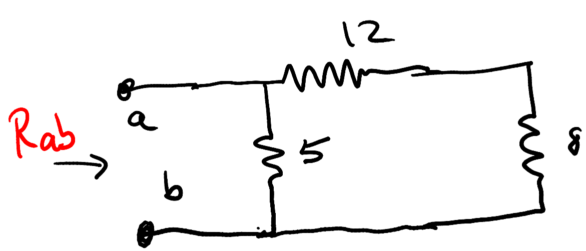


$$I_o = 5A\left(\frac{8}{12+8}\right)$$

$$= 2A$$

B) $t > 0$ → switch open, Find R_{ab} with L removed

(redraw circuit)



$$R_{ab} = 5 \,//\, (12+8)$$

$$= 4\Omega$$

c) $\tau = \frac{L}{R} = \frac{2}{4} = \frac{1}{2}\ s$

d) $i(t) = I_o e^{-t/\tau}$

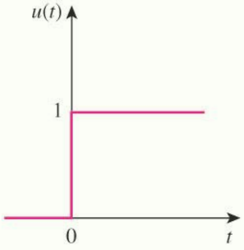
$$= 2e^{-2t}\ A$$

Answer: $i(t) = 2e^{-2t}\ A$

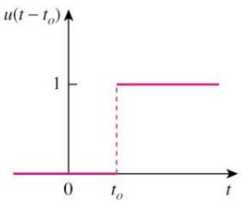
7.3 Unit-Step Function (1)

- The **unit step function** $u(t)$ is 0 for negative values of t and 1 for positive values of t.

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

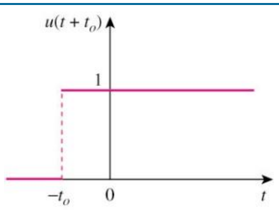


$$u(t - t_o) = \begin{cases} 0, & t < t_o \\ 1, & t > t_o \end{cases}$$



(a)

$$u(t + t_o) = \begin{cases} 0, & t < -t_o \\ 1, & t > -t_o \end{cases}$$

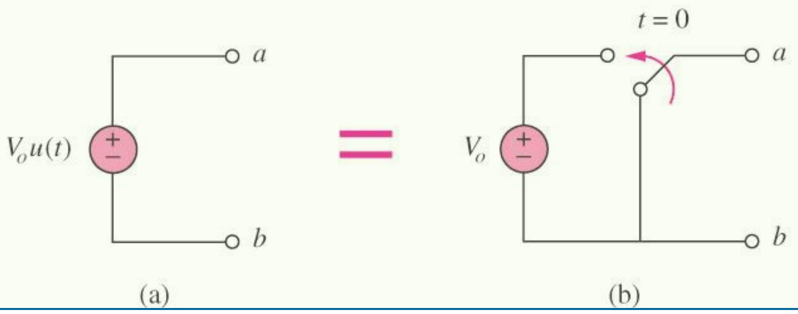


(b)

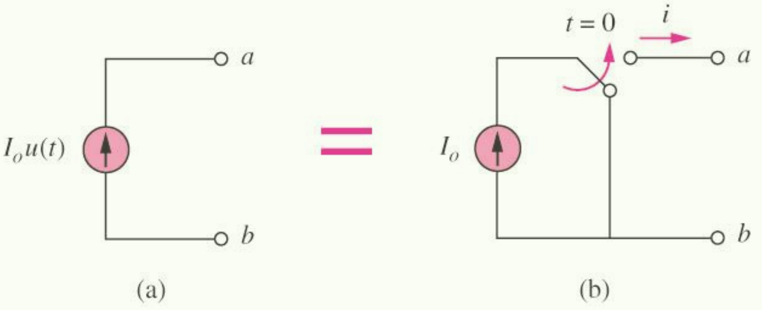
7.3 Unit-Step Function (2)

Represent an abrupt change for:

1. voltage source.

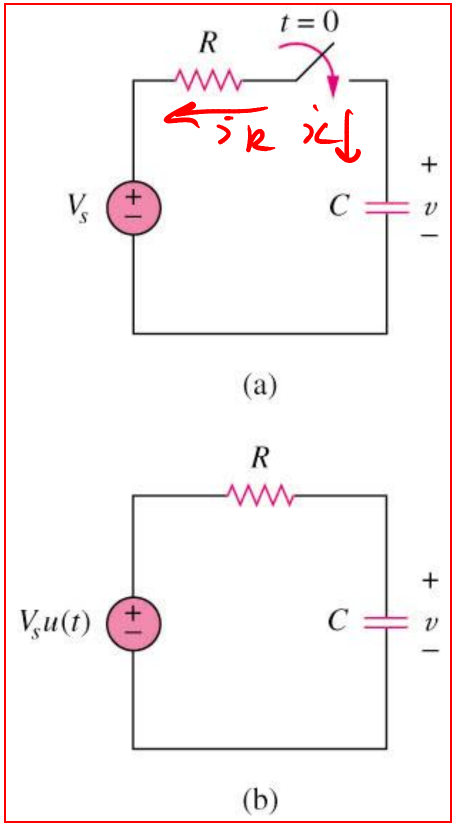


2. for current source:



7.4 The Step-Response of a RC Circuit (1)

- The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



- **Initial condition**:

$$v(0-) = v(0+) = V_0$$

- Applying KCL, $i_c + i_R = 0$

$$C\frac{dV}{dt} + \frac{(V - V_s u(t))}{R} = 0$$

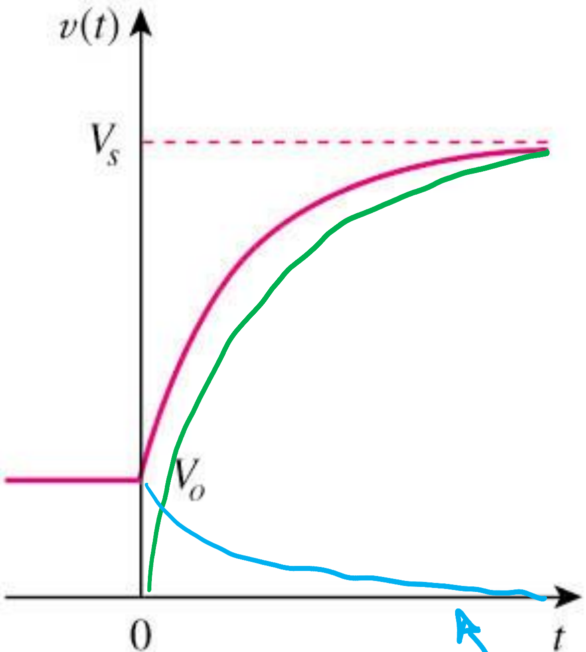
or

$$\frac{dv}{dt} = -\frac{v - V_s}{RC} u(t)$$

- Where u(t) is the unit-step function

7.4 The Step-Response of a RC Circuit (2)

- Integrating both sides and considering the initial conditions, the solution of the equation is:



$$v(t) = \begin{cases} V_0 & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau} & t > 0 \end{cases}$$

Final value at t -> ∞

Initial value at t = 0

Source-free Response

Complete Response = Natural response (stored energy) + Forced Response (independent source)

= $\boxed{V_0 e^{-t/\tau}}$ + $\boxed{V_s(1-e^{-t/\tau})}$

7.4 The Step-Response of a RC Circuit (3)

Three steps to find out the step response of an RC circuit:

1. The initial capacitor voltage v(0).
2. The final capacitor voltage v(∞) or, the final DC voltage across C.
3. The time constant τ.

$$v(t) = v(\infty) + [v(0+) - v(\infty)]e^{-t/\tau}$$

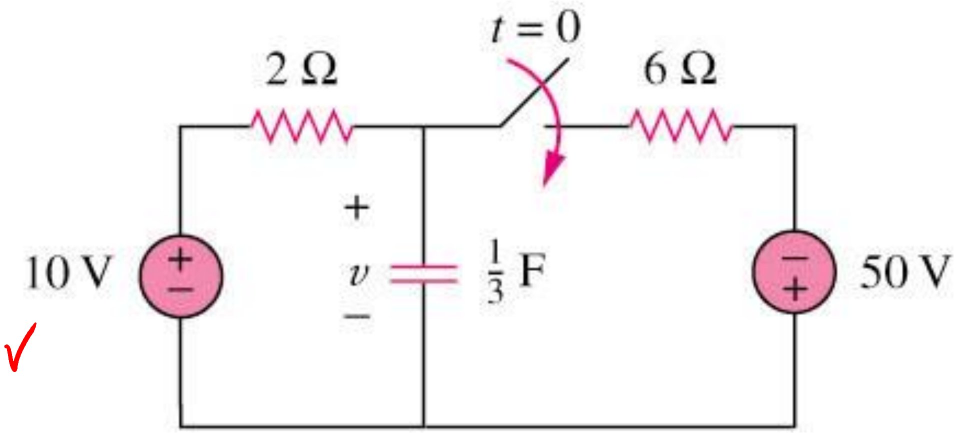
Note: The above method is a short-cut method. You may also determine the solution by setting up the circuit formula directly using KCL, KVL , ohms law, capacitor and inductor VI laws.

7.4 Step-Response of a RC Circuit (4)

Example 5

Find v(t) for t > 0 in the circuit in below. Assume the switch has been open for a long time and is closed at t = 0.

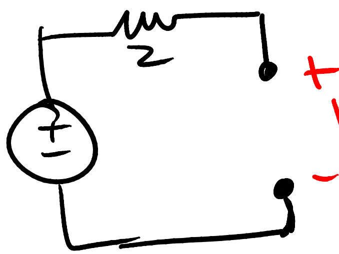
Calculate v(t) at t = 0.5.



1) $t < 0$

C is an open ckt

Find V_0

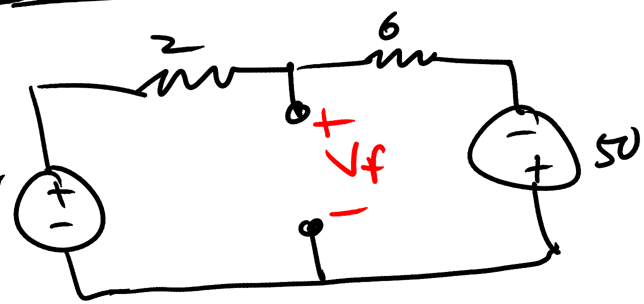


$V_0 = 10V$

2) as $t \to \infty$

C is an open ckt.

Find V_f



Nodal Analysis

$$\frac{V_f - 10}{2} + \frac{V_f - (-50)}{6} = 0$$

$$3V_f - 30 + V_f + 50 = 0$$

$$V_f = -5V$$

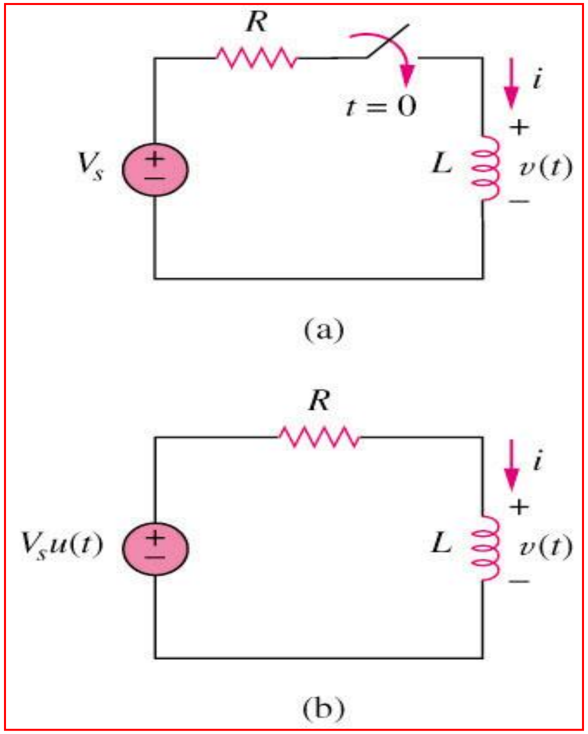
3) Find $\tau = RC = (2 \| 6)\frac{1}{3} = \frac{12}{24} = \frac{1}{2}$ s

4) $v(t) = V_f + (V_0 - V_f)e^{-t/\tau}$, $v(t) = -5 + (10 - -5)e^{-2t}$

Answer: $v(t) = 15e^{-2t} - 5$ *and v(0.5) = 0.5182V*

7.5 The Step-response of a RL Circuit (1)

- The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.



- **Initial current**

  $i(0-) = i(0+) = I_o$

- **Final inductor current**

  $i(\infty) = Vs/R$

- Time constant $\tau = L/R$

$$i(t) = \underbrace{\boxed{\frac{V_s}{R}}}_{} + (I_o - \frac{V_s}{R})e^{-\frac{t}{\tau}}u(t)$$

$I_f \quad I_o \quad I_f$

7.5 The Step-Response of a RL Circuit (2)

Three steps to find out the step response of an RL circuit:

1. The initial inductor current i(0) at t = 0+.
2. The final inductor current i(∞).
3. The time constant τ.

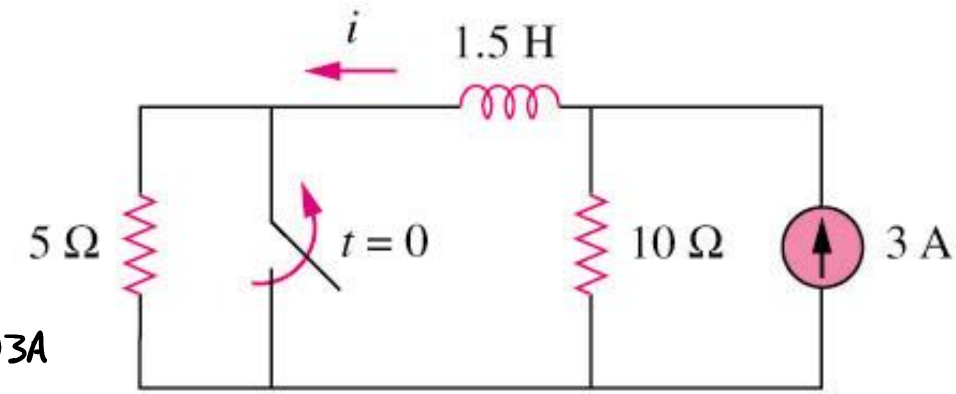
$$i(t) = i(\infty) + [i(0+) - i(\infty)]e^{-t/\tau}$$

Note: The above method is a short-cut method. You may also determine the solution by setting up the circuit formula directly using KCL, KVL , ohms law, capacitor and inductor VI laws.

Example 6

The switch in the circuit shown below has been closed for a long time. It opens at t = 0.

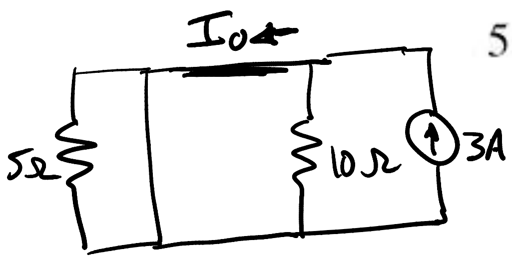
Find i(t) for t > 0.



1) $t<0$, L is short cct

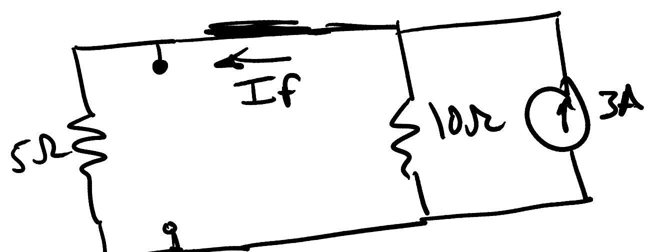
Find I_o

$I_o = 3A$



2) $t \to \infty$, L is a short cct

Find I_F



$I_f = 3\left(\frac{10}{5+10}\right) = 2A$

3) R_{eq} for $t>0$

$R = 5+10 = 15\Omega$

4) $\tau = \frac{L}{R} = 1.5/15 = 0.1\,s$

5) $i(t) = I_f + (I_o - I_f)e^{-t/\tau}$

$i(t) = 2 + (3-2)e^{-10t} =$

Answer: $i(t) = 2 + e^{-10t}$

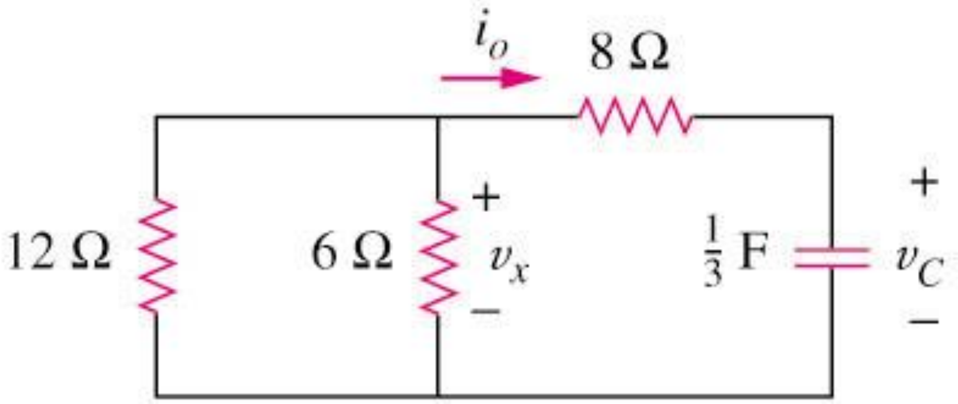
Handouts

7.1 Source-Free RC Circuit (4)

Example 1

Refer to the circuit below, determine v_C, v_x, and i_o for $t \geq 0$.

Assume that $v_C(0) = 30$ V.

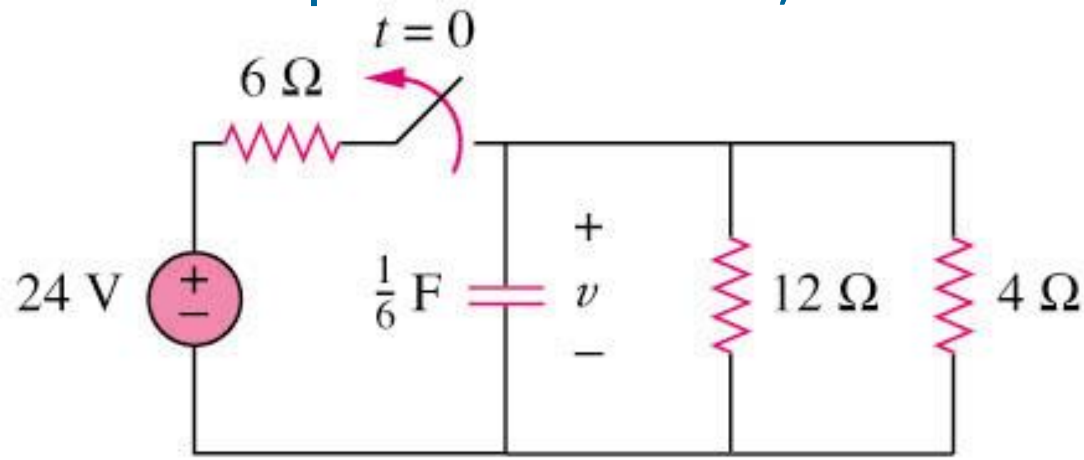


Answer: $v_C = 30e^{-0.25t}$ V ; $v_x = 10e^{-0.25t}$; $i_o = -2.5e^{-0.25t}$ A

7.1 Source-Free RC Circuit (5)

Example 2

The switch in circuit below is opened at t = 0, find v(t) for t ≥ 0.



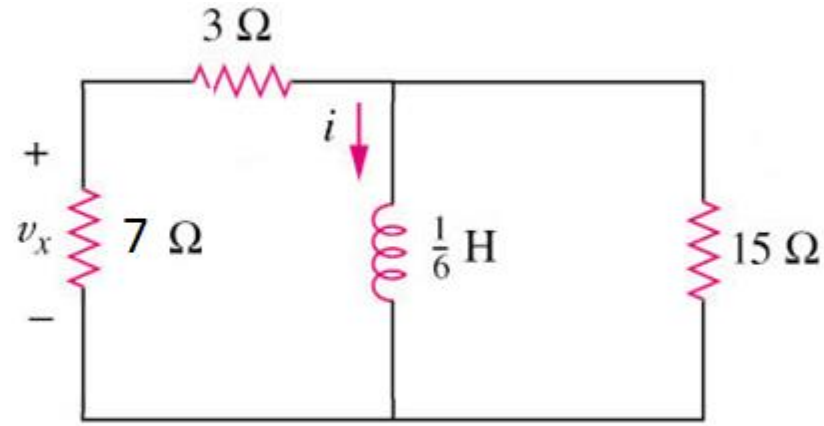
Answer: $V(t) = 8e^{-2t}\ V$

7.2 The Source-Free RL Circuit (5)

Example 3

Find i and v_x in the circuit.

Assume that i(0) = 5 A.

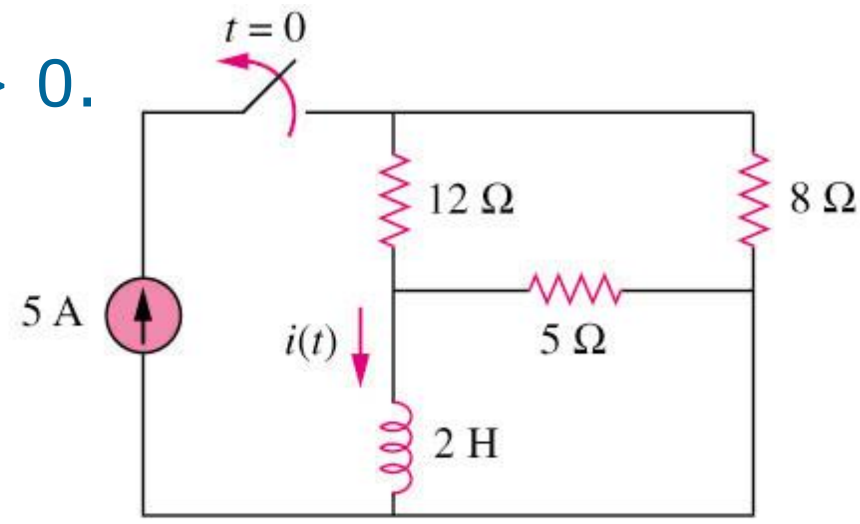


Answer: $i(t) = 5e^{-36t}$ A

7.2 The Source-Free RL Circuit (6)

Example 4

For the circuit, find i(t) for t > 0.



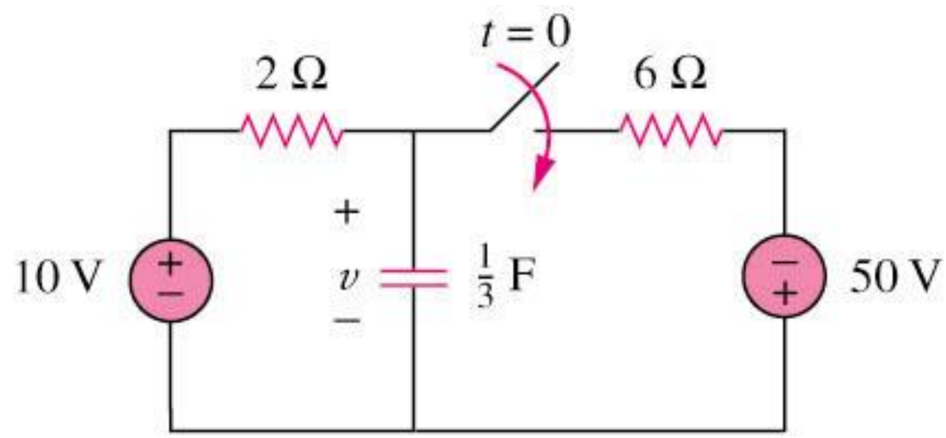
Answer: $i(t) = 2e^{-2t}\ A$

7.4 Step-Response of a RC Circuit (4)

Example 5

Find v(t) for t > 0 in the circuit in below. Assume the switch has been open for a long time and is closed at t = 0.

Calculate v(t) at t = 0.5.



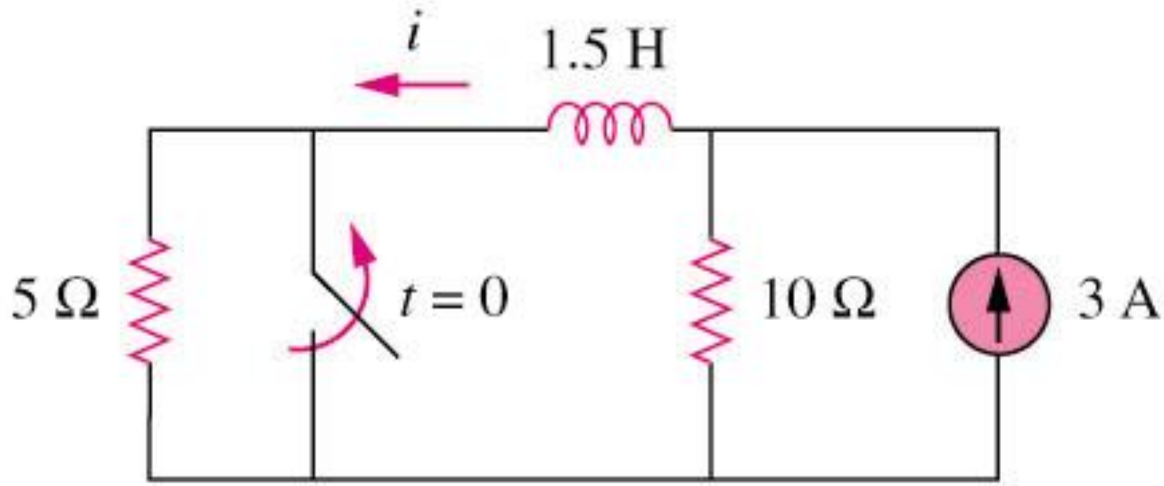
Answer: $v(t) = 15e^{-2t} - 5$ *and v(0.5) = 0.5182V*

7.5 The Step-Response of a RL Circuit (4)

Example 6

The switch in the circuit shown below has been closed for a long time. It opens at t = 0.

Find i(t) for t > 0.



Answer: $i(t) = 2 + e^{-10t}$