Chapter 7 Response of First-Order RL and RC Circuits

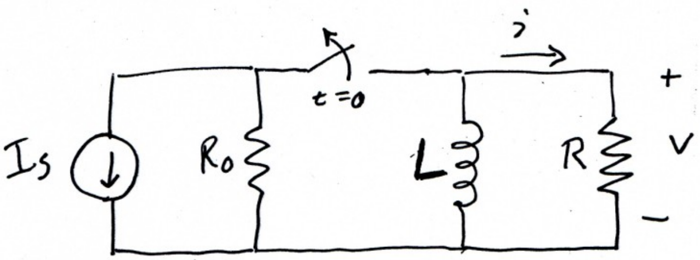
First-Order: circuit contains L's or C's <u>but not both</u>

Natural Response: initial stored energy (L or C) is dissipated into resistive network

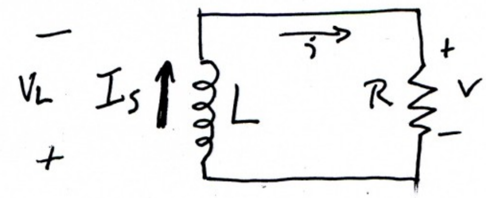
Step Response: A DC current or voltage is suddenly switched on and L or C acquires energy

7.1 Natural Response of RL Circuit

A typical RL circuit:



At t>0, Inductor has an initial current Io = Is, and the circuit reduces to:



$$V_L = L\frac{di}{dt}$$

Using KVL we can find and solve a differential equation for the current in the circuit

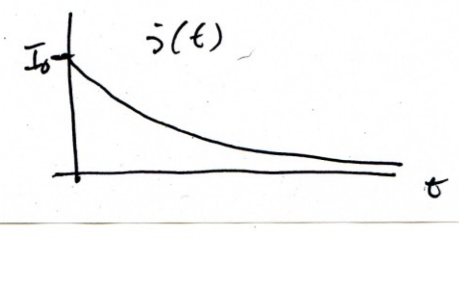
$$L\frac{di}{dt} + Ri = 0 \qquad (1^{st} \text{ order O.D.E.})$$

$$\frac{di}{dt}dt = -\frac{R}{L} i\, dt$$

$$\frac{di}{i} = -\frac{R}{L}dt \;, \qquad \int_{i(t_0)}^{i(t)} \frac{dx}{x} = -\frac{R}{L}\int_0^t dy \;,$$

$$\ln(x)\Big|_{i(t_0)}^{i(t)} = -\frac{R}{L}y\Big|_0^t \;, \qquad \ln\frac{i(t)}{i(t_0)} = -\frac{R}{L}t \;, \qquad \frac{i(t)}{i(t_0)} = e^{-\frac{R}{L}t}$$

$$\text{or,} \quad i(t) = i(t_0)e^{-\frac{R}{L}t}$$



Of particular interest is the point in time when the switch is opening. Remember that current in an inductor cannot experience a "step", therefore i(0-) = i(0+) = Io, which was established in our example circuit as Is, the current source current. Therefore,

$$i(t) = I_s e^{-\frac{R}{L}t} \qquad t \geq 0^+$$
$$i(t) = I_s \qquad t \leq 0^- \text{ (before the switch opened)}$$

The voltage across the Resistor, v(t), is now derived from Ohm's Law:

$$v(t) = R\,i(t) = \begin{cases} I_s R e^{-\frac{R}{L}t} & t \geq 0^+ \\ 0 & t \leq 0^- \text{ (before switch opened)} \end{cases}$$

And unlike the current, the voltage across an inductor can experience a step, and since i(t) is constant for t<0, then v(t) = 0 before the switch opens.

At t = 0, the voltage is undefined because of the slope discontinuity in i. So to be precise, v(t) is defined for times smaller than 0- , i.e. the instant before the switch opens, and for times greater than 0+, the instant after the switch opens.

$$v(0^+) = I_s R$$
$$v(0^-) = 0$$

Once we derive current and voltage, the power and energy can now be determined. Continuing with the power dissipated in the resistor:

$$p = vi = i^2 R = \frac{v^2}{R} \text{ any of which produces } p = I_s^2 R e^{-2\left(\frac{R}{L}\right)t}, \; t \geq 0^+$$

To calculate the energy dissipated in the resistor (not stored in the inductor) we need to integrate power:

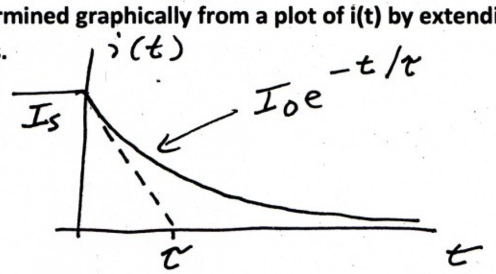
$$w = \int_0^t p\,dx = \int_0^t I_s^2 R e^{-2\left(\frac{R}{L}\right)x} dx = \frac{1}{2\left(\frac{R}{L}\right)} I_s^2 R \left(1 - e^{-2\left(\frac{R}{L}\right)t}\right)$$
$$= \frac{1}{2} L I_s^2 \left(1 - e^{-2(R/L)t}\right), \; t \geq 0$$

Time Constant:

In the solution for i(t), the coefficient of t in the exponent , R/L , plays a key factor in determining the rate of decay of i(t) or the rate of dissipation of energy into the resistor. The reciprocal of this coefficient is referred to as

$$\tau = \text{time constant} = \frac{L}{R}$$

And it can be determined graphically from a plot of i(t) by extending a tangent to the curve at t=0 to the point where it crosses the t axis.



SUMMARY: the GENERAL SOLUTION to the natural response boils down to 3 basic steps.

1) finding initial current i(0) in the inductor
2) Find the time constant
3) The solution for the current thru the inductor is $i(t) = i(0) e^{-t/\tau}$
4) Use this current to derive v, p, w, and any other circuit variables

EXAMPLE 5.6

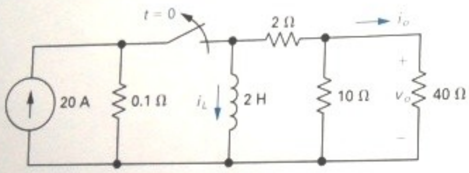


Figure 5.25 The circuit for Example 5.6.

The switch in the circuit shown in Fig. 5.25 has been closed for a long time before it is opened at $t = 0$. Find

(a) $i_L(t)$ for $t \geq 0$

(b) $i_o(t)$ for $t \geq 0^+$

(c) $v_o(t)$ for $t \geq 0^+$

(d) the percentage of the total energy stored in the 2 H inductor that is dissipated in the 10 Ω resistor

5.6 The switch in the circuit shown has been closed for a long time and is opened at $t = 0$.

(a) Calculate the initial value of i.

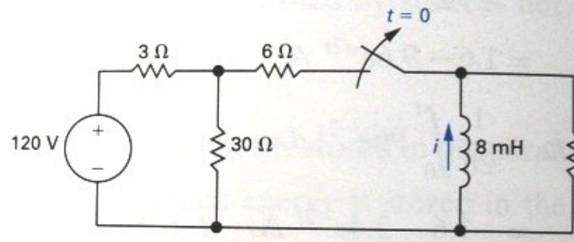
(b) Calculate the initial energy stored in the inductor.

(c) What is the time constant of the circuit for $t > 0$?

(d) What is the numerical expression for $i(t)$ for $t \geq 0$?

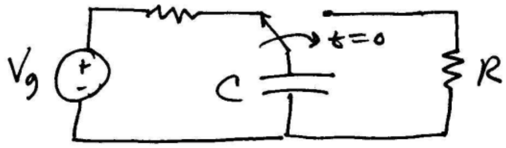
(e) What percentage of the initial energy stored has been dissipated in the 2 Ω resistor 5 ms after the switch has been opened?

ANSWER: (a) −12.5 A; (b) 625 mJ; (c) 4 ms; (d) $-12.5e^{-250t}$ A, $t \geq 0$; (e) 91.8%.

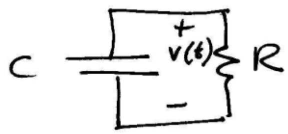


7.2 Natural Response of an RC Circuit

Very similar to RL circuit, but Capacitors look like open circuits in the presence of constant voltage (i goes to 0), whereas Inductors look like short circuits in the presence of constant voltage (v goes to 0). A typical RC Circuit:



This circuit is used to charge the capacitor with an initial voltage. The switch at t=0 isolates the right portion, which simplifies the analysis to:



where $v(0) = V_g$

By following the same steps as were used for the RL circuit, the general solution is

(KCL at node +)

$$C\frac{dv}{dt} + \frac{v}{R} = 0 \longrightarrow v(t) = v(0)\, e^{-t/RC}, \quad t \geq 0$$

In the example circuit, the initial voltage is Vg, and because the capacitor cannot allow a step in voltage,

$$v(0^-) = v(0) = v(0^+)$$
$$= V_g$$

The time constant is now the product of the capacitance and the thevenin resistance of the circuit seen by the capacitor: $\tau = RC$

As in the RL circuit, once we determine v(t) we can then derive i, p and w.

related to Resistor R

$$i(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau} \qquad t \geq 0^+$$

$$p(t) = v \cdot i = \frac{V_0^2}{R} e^{-2t/\tau} \qquad t \geq 0^+$$

$$w(t) = \int_0^t p\, dx = \int_0^t \frac{V_0^2}{R} e^{-2t/\tau}\, dx = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}), \quad t \geq 0$$

Summarizing, the method for finding the natural response of an RC circuit is:

1) Find initial voltage v(0) across capacitor
2) Find the time constant of the circuit
3) Use the equation $v(t) = V_o\ e^{-t/RC}$
4) Use the solution for v(t) to derive all other related varibles.

5.8 RC Natural Response

The switch in the circuit shown has been closed for a long time and is opened at $t = 0$. Find

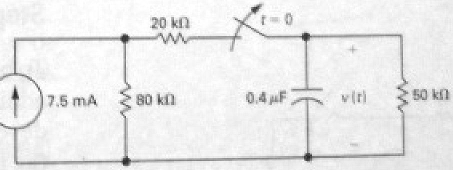
(a) the initial value of $v(t)$

(b) the time constant for $t > 0$

(c) the numerical expression for $v(t)$ after the switch has been opened

(d) the initial energy stored in the capacitor

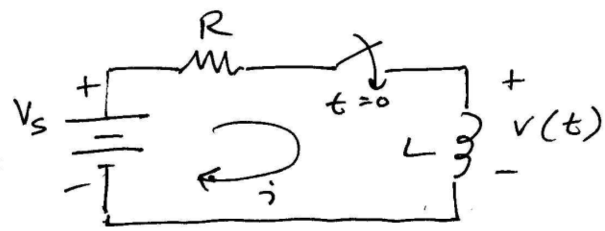
(e) the length of time required to dissipate 75% of the initially stored energy.



ANSWER: (a) 200 V; (b) 20 ms; (c) $200e^{-50t}$ V, $t \geq 0$; (d) 8 mJ; (e) 13.86 ms.

7.3 Step Response of RL and RC circuits

A typical RL step response circuit is shown here:



The difference from the natural response is that we are now energizing the inductor from some initial state by applying a DC voltage across it for t>=0. As in the prior examples, we can derive the equation for the step response solution using circuit analysis, which is slightly more complicated than the natural response.

$$KVL: \quad V_s = Ri + L\frac{di}{dt}, \quad di = -\frac{R}{L}\left(i - \frac{V_s}{R}\right)dt$$

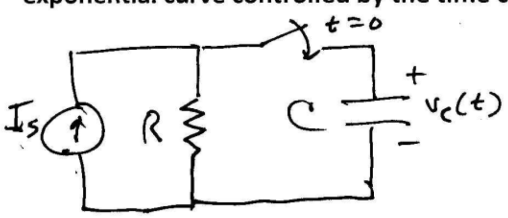
$$\int_{I_0}^{i(t)} \frac{dx}{x-(V_s/R)} = -\frac{R}{L}\int_0^t dy, \quad \ldots \quad i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-(R/L)t}$$

The result is that the inductor current i(t) transitions from its initial value (when the inductor behaves like a short circuit) to its final value (when it is again a short with a different source driving it) with an exponential curve controlled by the time constant of the circuit the inductor is attached to.

$$i(t) = I_f + [I_0 - I_f]e^{-(R/L)t}$$

where I_0 = initial current at $t = 0^-$
I_f = final current at $t = \infty$

The same thing happens with an RC circuit step response. The capacitor voltage v(t) transitions from its initial value (when it is an open circuit) to its final value (when it is again an open circuit), with an exponential curve controlled by the time constant of the circuit the inductor is attached to.



$$KCL: \quad C\frac{dv_c}{dt} + \frac{v_c}{R} = I_s, \quad \frac{dv_c}{dt} + \frac{v_c}{RC} = \frac{I_s}{C}$$

$$V_c(t) = I_s R + (V_0 - I_s R)e^{-t/RC}, \quad t \geq 0$$

$$= V_f + (V_0 - V_f)e^{-t/RC}$$

Summarizing, the General Method for finding the step response of an RL or RC circuit is:

1) **Identify the variable of interest for the circuit. For RL circuits, it's the inductor current, for RC circuits, the capacitor voltage.** Here we call it $x(t)$
2) **Determine the initial value of the variable of interest, which is its value at t0.** (X_0)
3) **Determine the final value of the variable of interest.** (X_f)
4) **Determine the time constant for the circuit.** (τ)
5) **The solution for the variable of interest is then:**

$$x(t) = X_f + (X_0 - X_f)e^{-t/\tau}$$

5.10 Step response of an RL circuit

The switch in the circuit shown in Fig. 5.37 has been in position a for a long time. At $t = 0$, the switch moves from position a to position b. The switch is a make-before-break type; that is, the connection at position b is established before the connection at position a is broken, so there is no interruption of current through the inductor.

(a) Find the expression for $i(t)$ for $t \geq 0$.

(b) What is the initial voltage across the inductor just after the switch has been moved to position b?

(c) Does this initial voltage make sense in terms of circuit behavior?

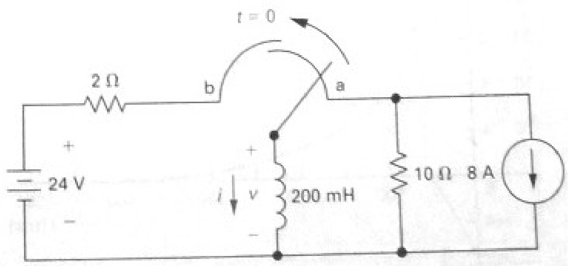


Figure 5.37 The circuit for Example 5.10.

5.11 Step Response of an RC Circuit

The switch in the circuit shown in Fig. 5.40 has been in position 1 for a long time. At $t = 0$, the switch moves to position 2. Find

(a) $v_o(t)$ for $t \geq 0$

(b) $i_o(t)$ for $t \geq 0^+$