Circuit Theory

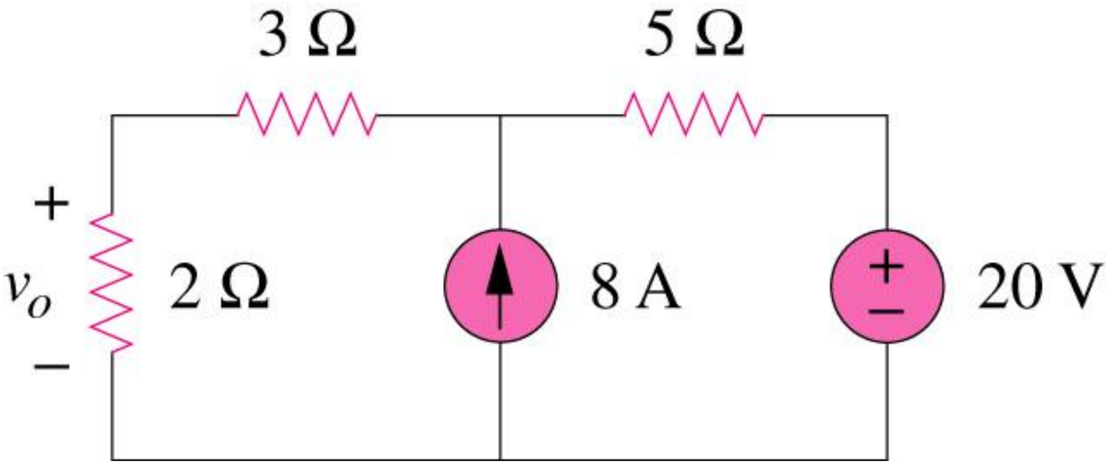
Chapter 4

Circuit Theorems

Circuit Theorems - Chapter 4

4.1 Motivation (1)

If you are given the following circuit, are there any other alternative(s) to determine the voltage across 2Ω resistor?



What are they? And how?

Can you work it out by inspection?

4.2 Linearity Property (1)

It is the property of an element describing a linear relationship between cause and effect.

A linear circuit is one whose output is linearly related (or directly proportional) to its input.

Homogeneity (scaling) property

$$\underset{\text{Output}}{\boldsymbol{v}} = \overset{\text{Input}}{\boldsymbol{i}}\, \boldsymbol{R} \quad \longrightarrow \quad \underbrace{\boldsymbol{k\,v}}_{\text{scaled output}} = \overbrace{\boldsymbol{k\,i}}^{\text{scaled input}}\, \boldsymbol{R}$$

Additive property

$$\boldsymbol{v_1} = \overset{\text{Input1}}{\boldsymbol{i_1}}\, \boldsymbol{R} \text{ and } \boldsymbol{v_2} = \overset{\text{Input2}}{\boldsymbol{i_2}}\, \boldsymbol{R}$$

$$\longrightarrow \quad \boldsymbol{v} = (\boldsymbol{i_1} + \boldsymbol{i_2})\, \boldsymbol{R} = \underbrace{\boldsymbol{v_1} + \boldsymbol{v_2}}_{\text{Output is sum of individual outputs}}$$

4.3 Superposition Theorem (1)

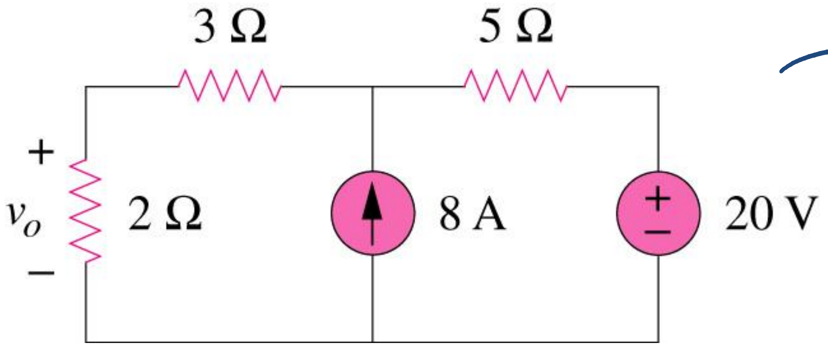
A gift of the Additive property of Linearity.

It states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltage across (or currents through) that element due to EACH independent source acting alone.

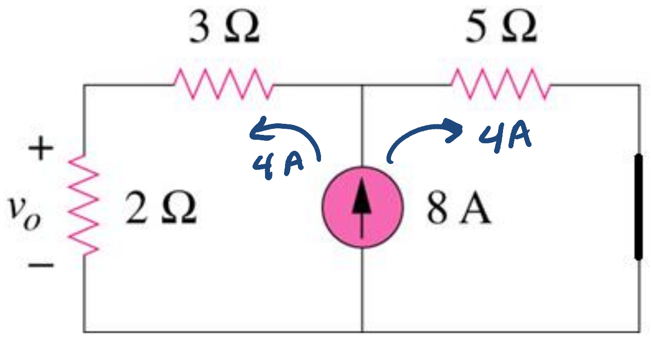
The principle of superposition helps us to analyze a linear circuit with more than one independent source by calculating the contribution of each independent source separately.

4.3 Superposition Theorem (2)

We consider the effects of 8A and 20V one by one, then add the two effects together for final v_o.

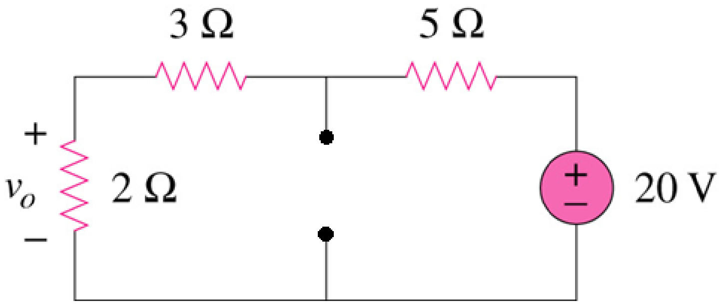


Effect of 8A SRC



$v_o = 4(2) = 8$

Effect of 20V SRC



$v_o = \left(\frac{2}{2+3+5}\right)20 = 4$

Total $v_o = 8 + 4 = \underline{\underline{12V}}$

4.3 Superposition Theorem (3)

Steps to apply superposition principle

1. Turn off all independent sources except one source. Find the output (voltage or current) due to that active source using nodal or mesh analysis.

2. Repeat step 1 for each of the other independent sources.

3. Find the total contribution by adding algebraically all the contributions due to the independent sources.

4.3 Superposition Theorem (4)

Two things have to be keep in mind:

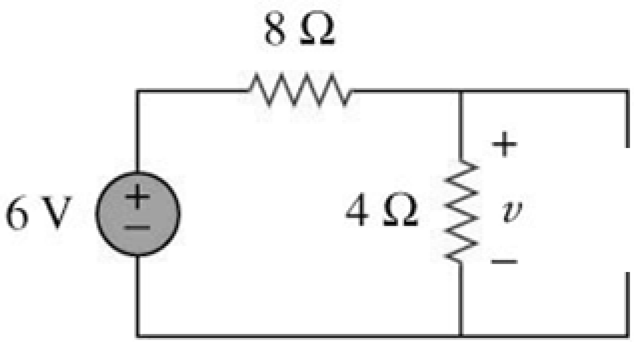
1. When we say turn off all other independent sources:
   - Independent voltage sources are replaced by 0 V (short circuit) and
   - Independent current sources are replaced by 0 A (open circuit).

2. Dependent sources are left intact because they are controlled by circuit variables.

4.3 Superposition Theorem (5)

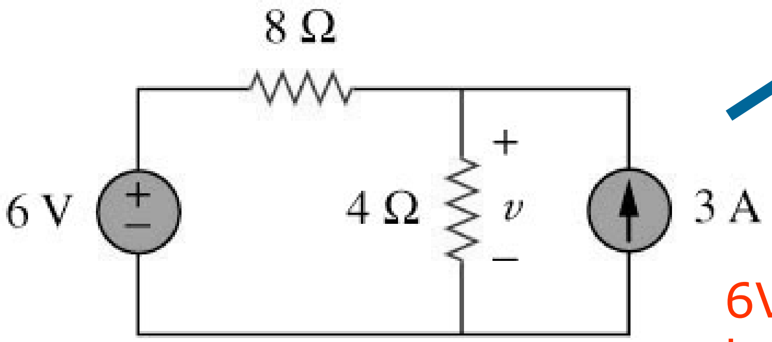
Example 2

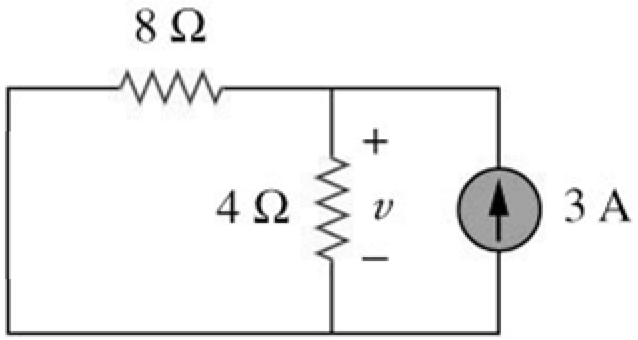
Use the superposition theorem to find v in the circuit shown below.



$v = \left(\frac{4}{4+8}\right)6 = 2v$

3A is turned off by open-circuit





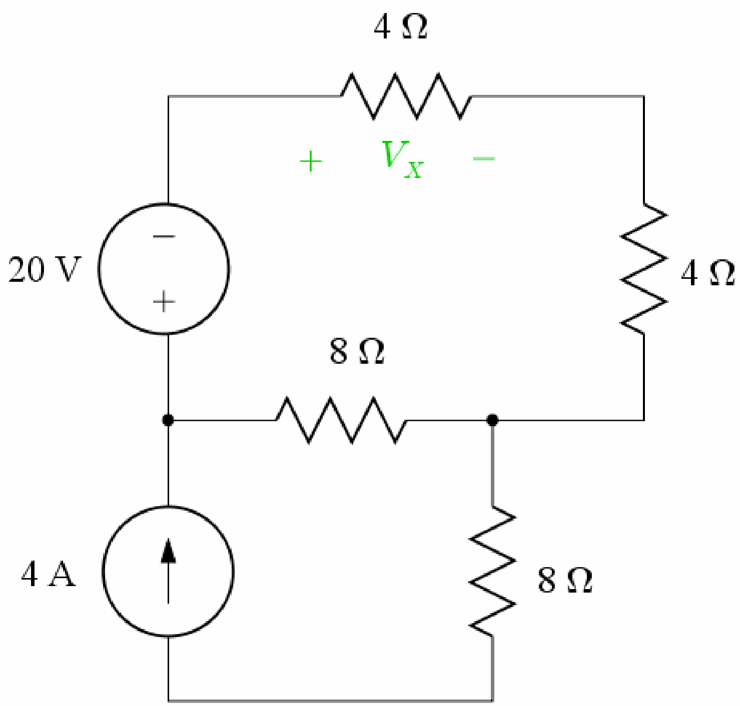
6V is turned off by short-circuit

$v = 3(4 \| 8) = 3\left(\frac{32}{12}\right)$

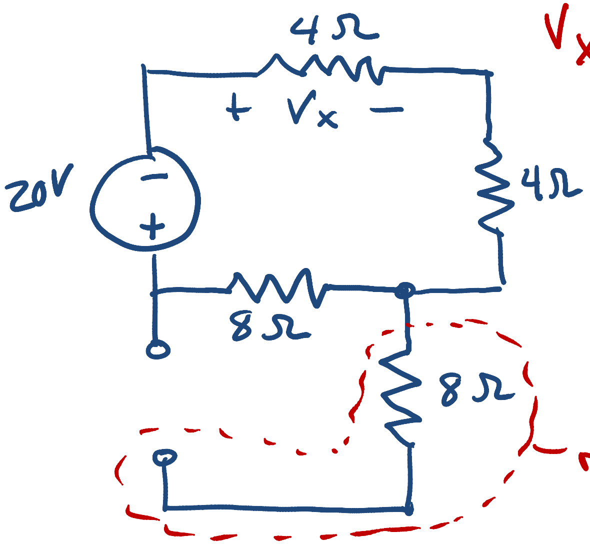
$= 8v$

answer *v* = 10V $= 2v + 8V$

1) Use Superposition to find Vx (3V)



a) Turn off 4A src

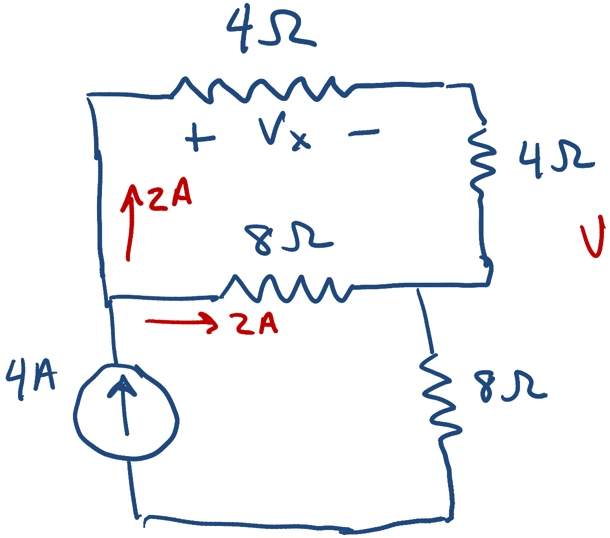


$$V_x = \left(\frac{-4}{4+4+8}\right)(20)$$

$$= -5V$$

remove, no current

b) Turn off 20V src



$$V_x = 2(4)$$

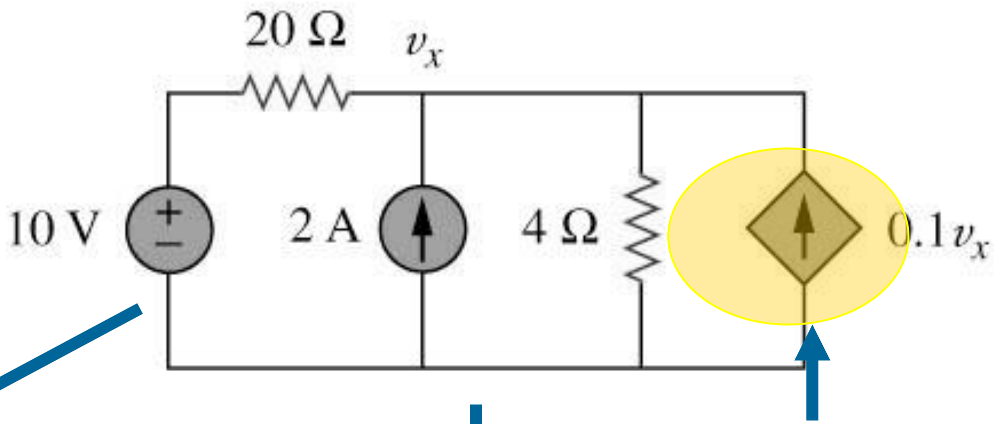
$$= 8V$$

Total $V_x = 8 - 5 = \underline{3V}$

4.3 Dependent Sources are not turned off when using the Superposition Theorem

Example 3

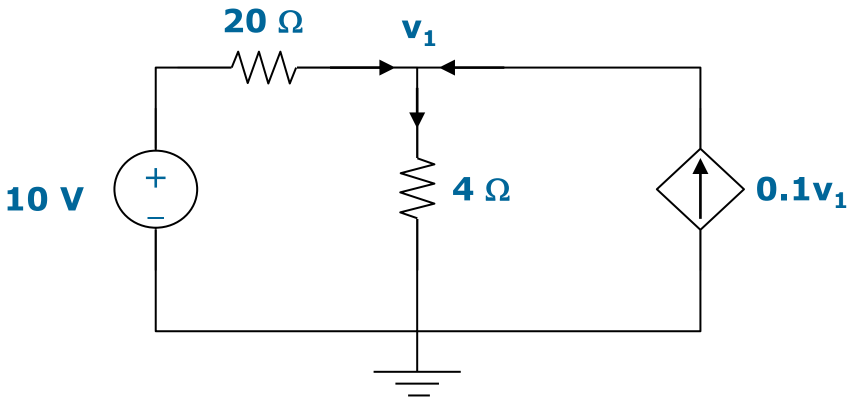
Use superposition to find v_x in the circuit below.



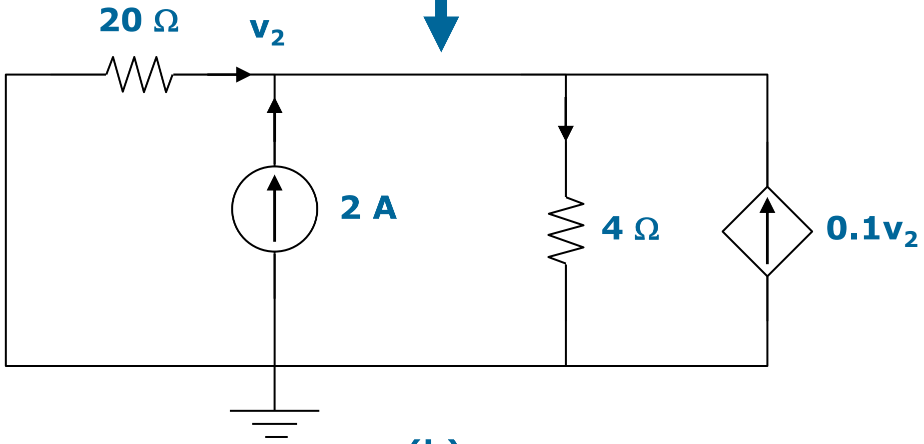
2A is discarded by open-circuit

10V is discarded by open-circuit

Dependant source keep unchanged



(a)



(b)

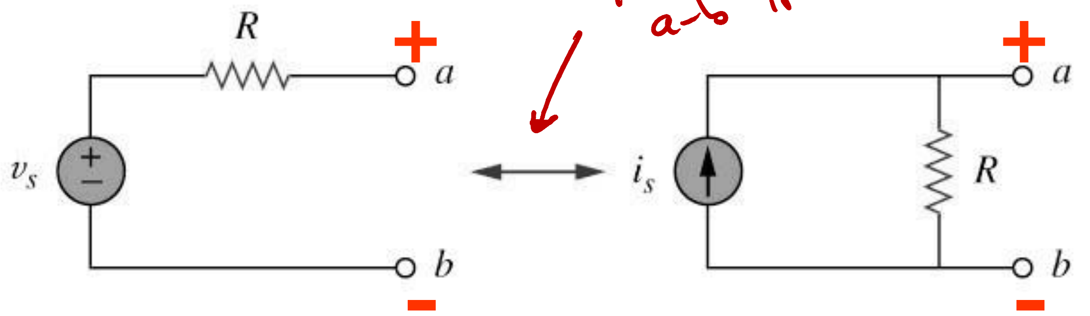
*Vx = 12.5V

4.4 Source Transformation (1)

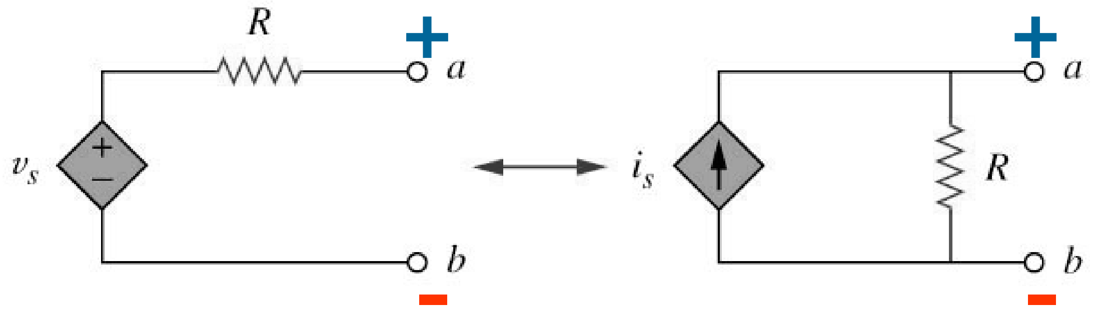
- An equivalent circuit is one whose *v-i* characteristics at some terminals (a-b) are identical with the original circuit.
- Source Transformation is the process of replacing a voltage source v_S in series with a resistor *R* by a current source i_S in parallel with a resistor *R*, or vice versa.
- The value of i_s or v_s in the replacement circuit is chosen to have the same behavior at the terminals of interest.

4.4 Source Transformation (2)

Equivalent at a-b if $i_s = V_s/R$



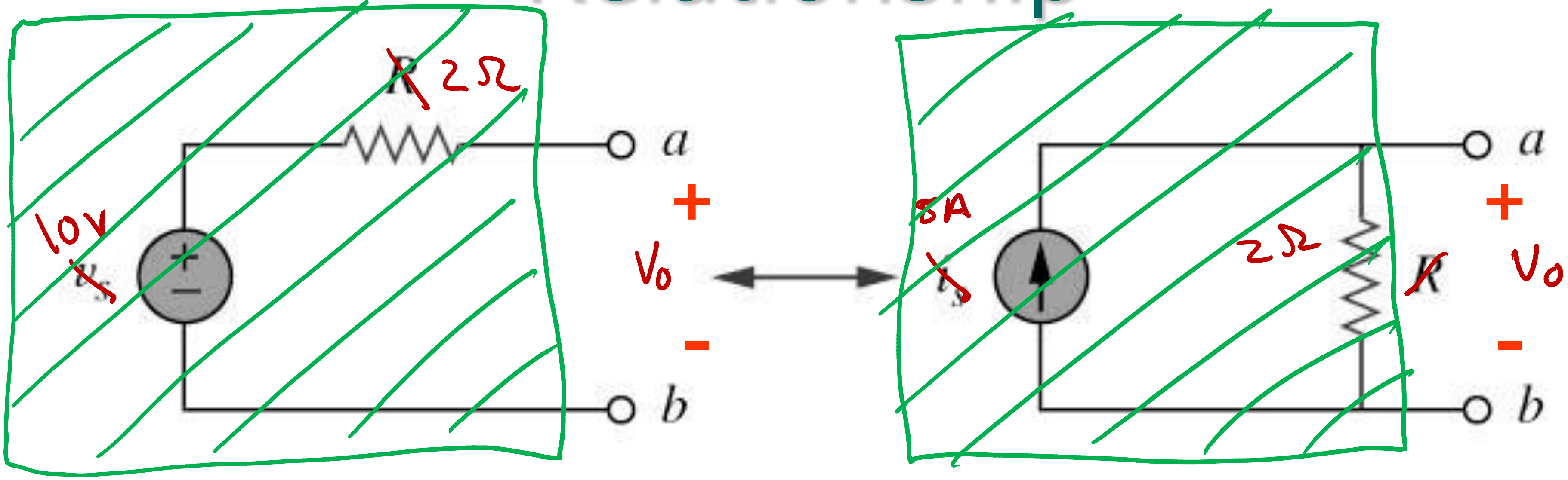
(a) Independent source transform



(b) Dependent source transform

- The arrow of the current source is directed toward the positive terminal of the voltage source.
- The source transformation is not possible when R = 0 for voltage source and R = ∞ for current source.

4.4 Source Transformation Relationship



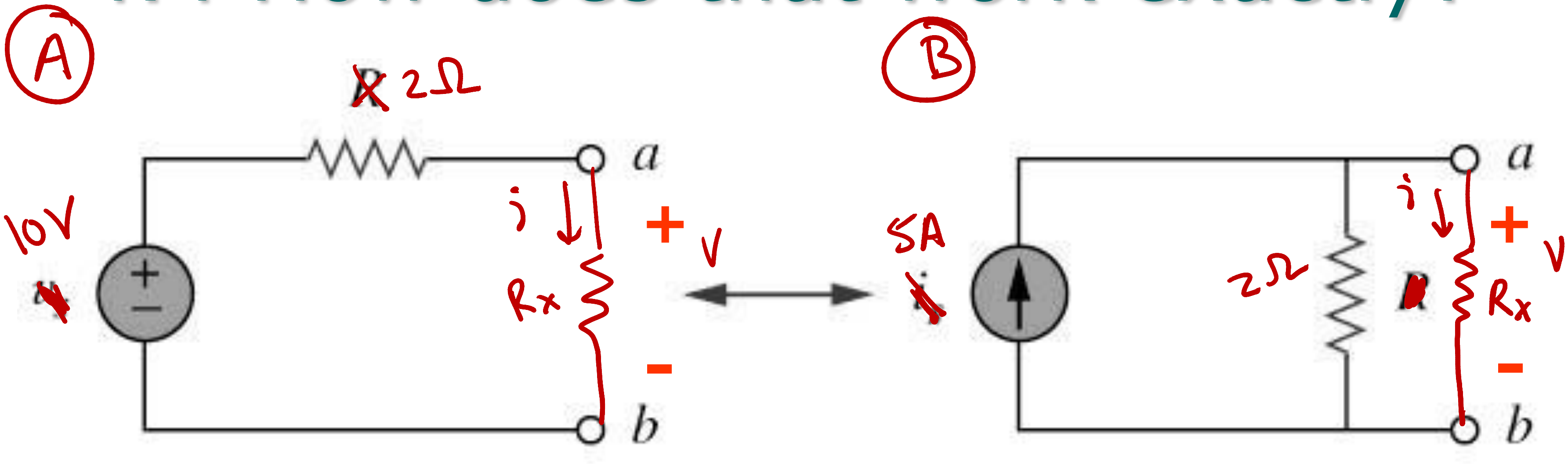
- The two circuits are equivalent **at a-b**
  - IF **vs = is R** or **is = vs/R**

- For example, if vs = 10V and R = 2 Ohm
  - then is = 5A in source transform
  - **same effect at a-b !**

V_o = 10 V for both ccts !

If we hide the ccts, we cannot tell which is which by measuring V or i at a-b

4.4 How does that work exactly?



- **Let's add a resistor R_x at a-b in both circuits**

Ⓐ $v = \left(\frac{R_x}{R_x+2}\right) 10\ V$, $i = \frac{10}{R_x+2}$ A

Ⓑ $v = 5\left(R_x \| 2\right)$, $i = \left(\frac{2}{R_x+2}\right)5$ A

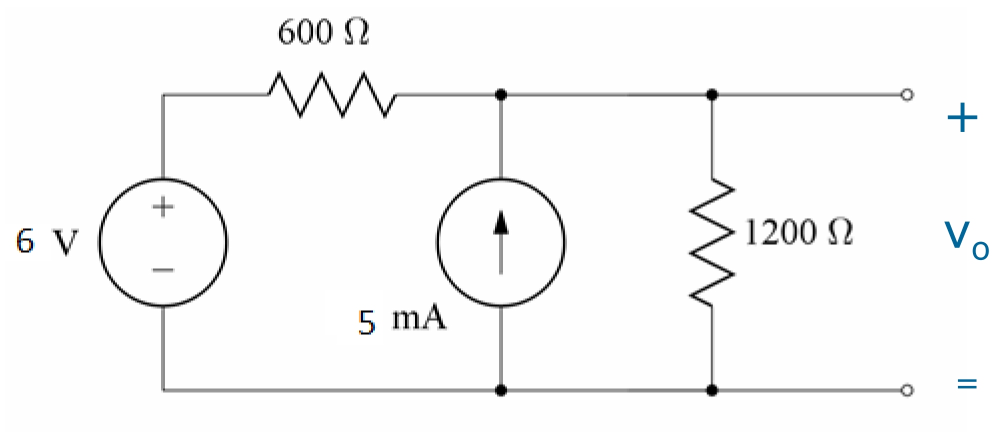
$= 5\left(\frac{2R_x}{R_x+2}\right)$

∴ Same behavior for all R_x !!

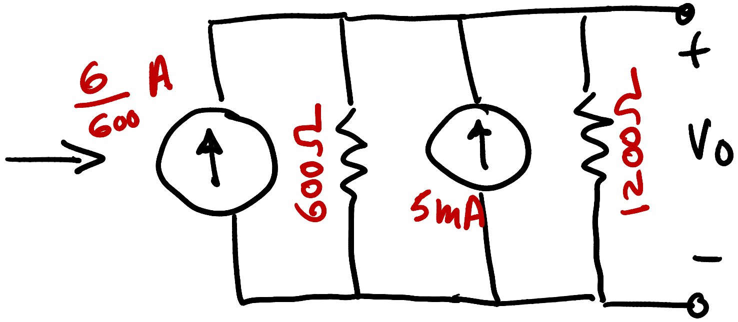
4.4 Source Transformation (3)

Example 4

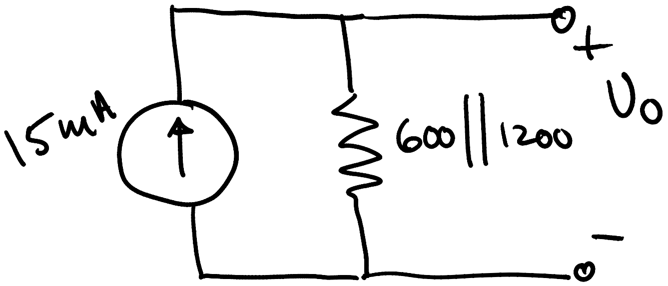
Find v_o in the circuit shown below using source transformation.



a) Xform V → I Src



b) Combine parallel R's & I-SRCs

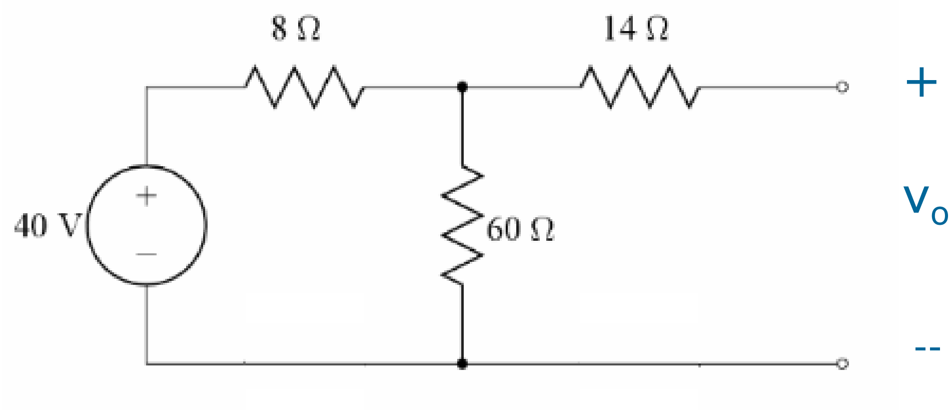


c) Calc Vo

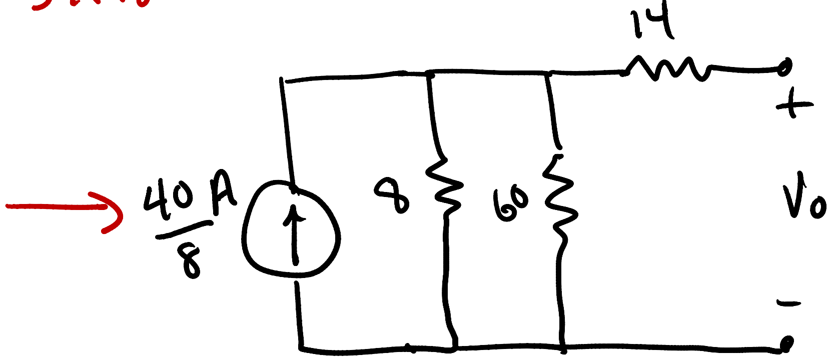
$$V_o = (.015)\left(\frac{600\cdot 1200}{1800}\right)$$

$$= \underline{\underline{6\ V}}$$

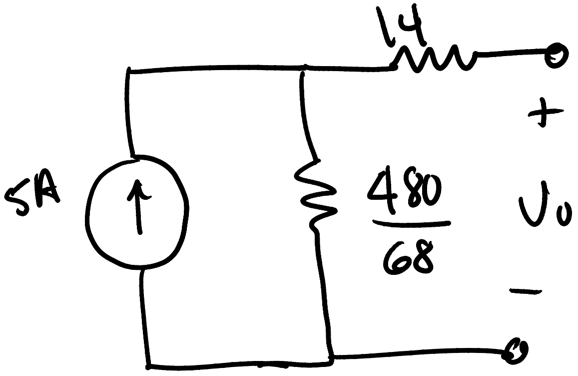
2) Find v_o in the circuit shown below using source transformation.



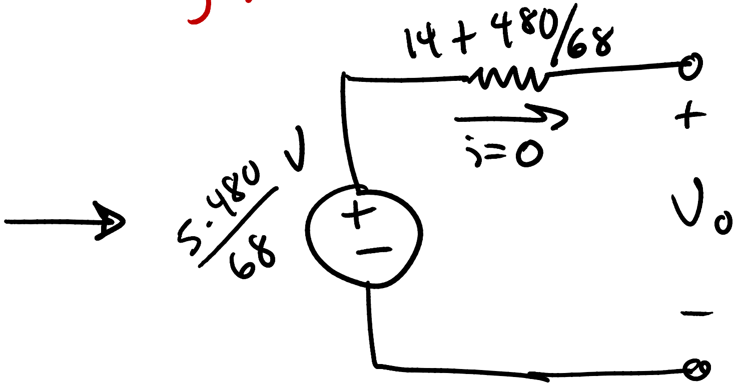
a) Xform V → I src



b) Combine || R's



c) X form I → V src

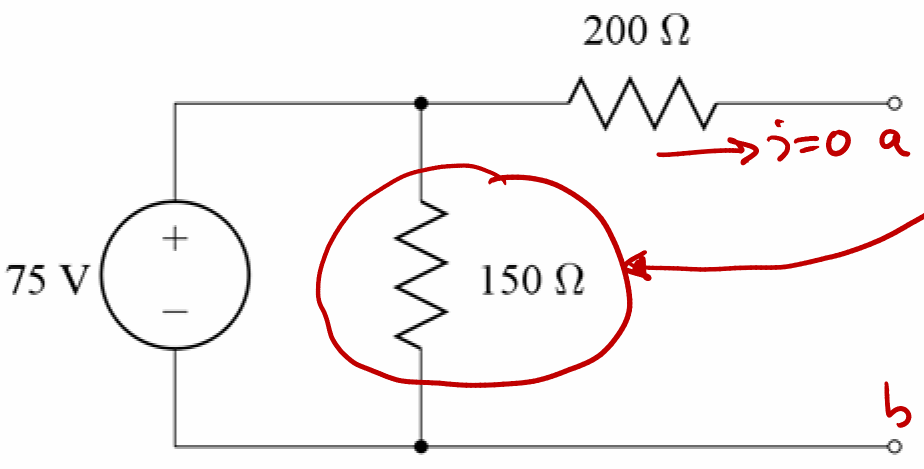


d) Calc v_o

$$v_o = \frac{5 \cdot 480}{68}$$

$$v_o = \underline{\underline{35.29\ V}}$$

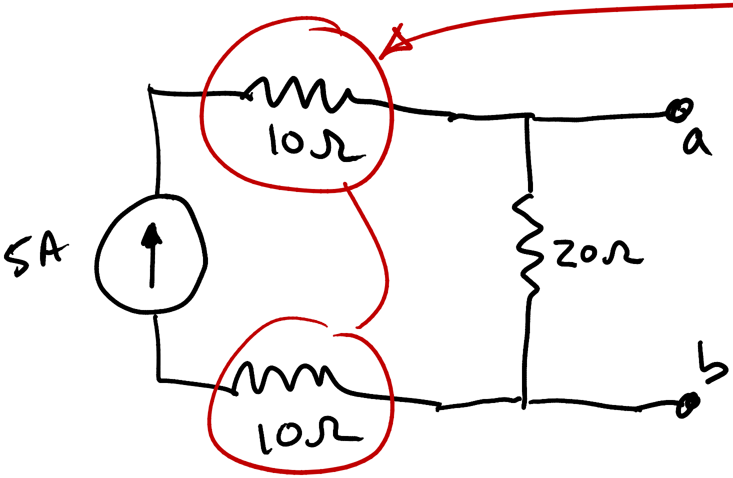
Some other tricks with Source Transform:
Resistors in Parallel with V source can be removed
Resistors in Series with I source can be removed



Can Remove. Does not affect V_{ab}.

(Does draw more current from V-src)

$V_{ab} = 75\,v$ regardless



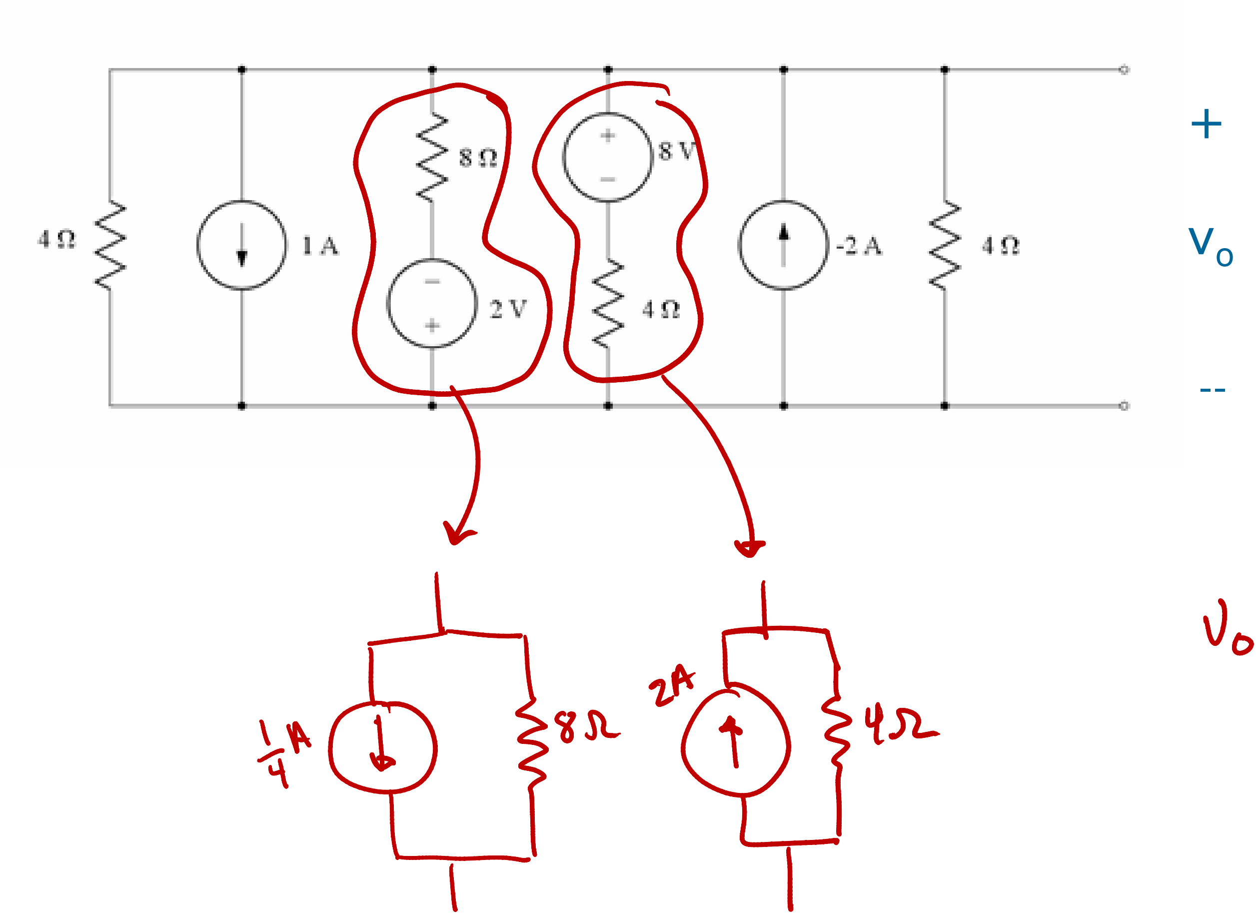
Can remove. Does not affect current flowing thru 20Ω Resistor.

(Does require more voltage from I-src)

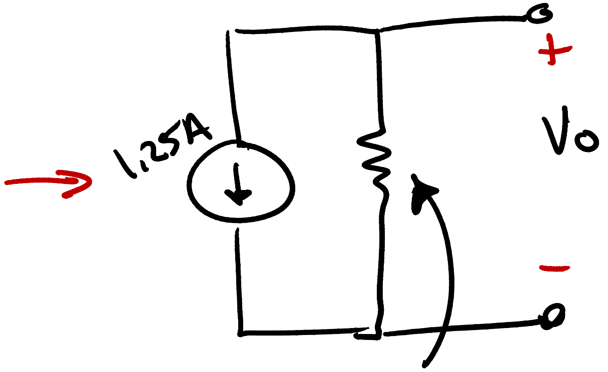
$V_{ab} = 100\,v$ regardless

Multiple sources in parallel

can be converted to current sources, added and reduced
Likewise, sources in series → Voltage sources, added, reduced



Combine I-srcs + || R's



$R = 4 \| 8 \| 4 \| 4$

$$V_o = -1.25\left(\frac{32}{12} \| 2\right)$$

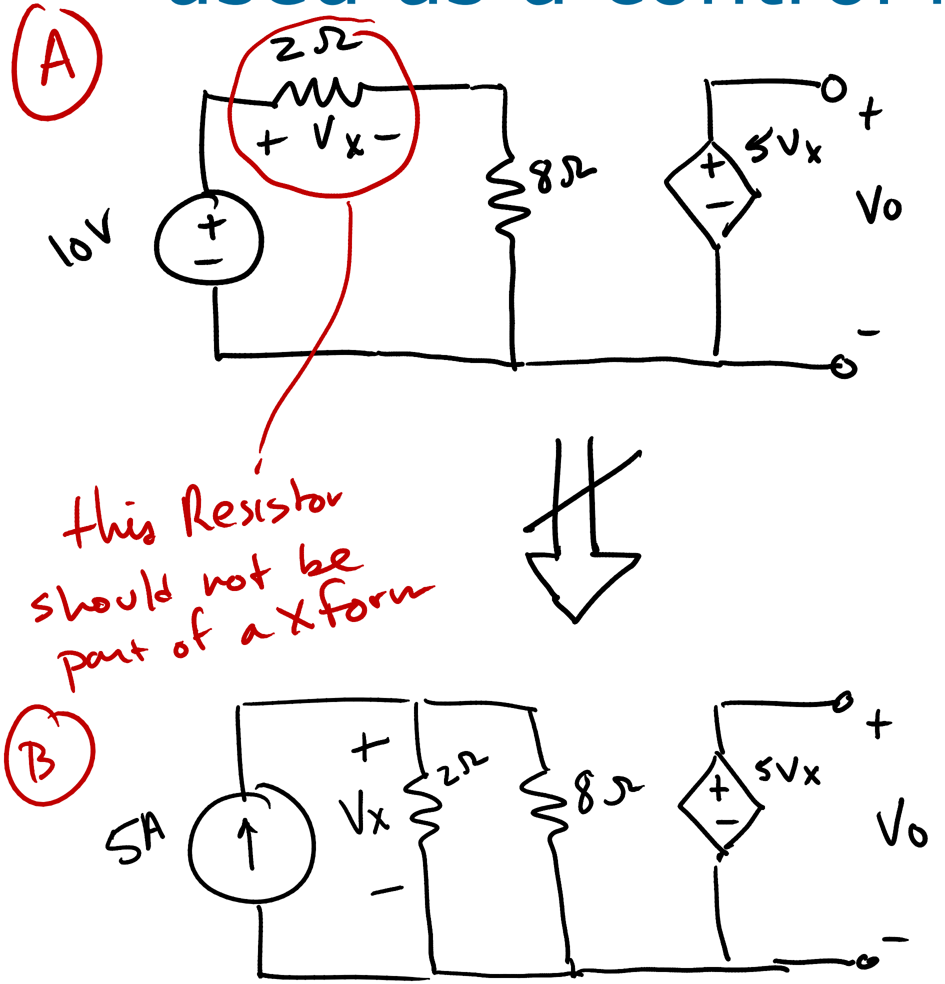
$$= -1.25\left(\frac{\frac{16}{3}}{2+\frac{8}{3}}\right)$$

$$= -1.25\left(\frac{16}{14}\right) = \frac{-40}{7}$$

$$= -5.71 \text{ V}$$

However, Beware "Rebold's Rule"

- Never Source Transform a Resistor used as a control for a Dependent Src



this Resistor should not be part of a xform

$V_x = 10\left(\frac{2}{10}\right) = 2V$

$V_o = 5V_x = \boxed{10V}$

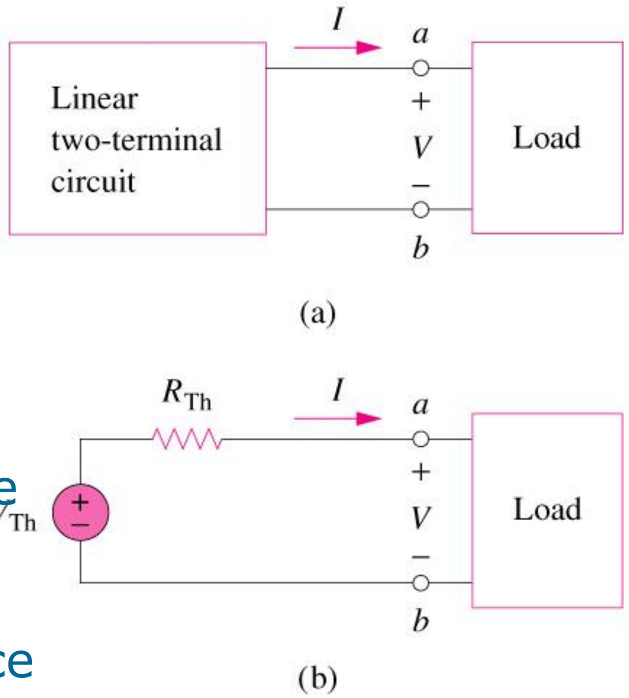
This Source Xform has changed a distant cct Variable

$V_x = 5(2\|8) = 5\cdot\frac{16}{10} = 8V$

$V_o = 5V_x = \boxed{40V}$!!

4.5 Thevenin's Theorem (1)

It states that a linear two-terminal circuit (Fig. a) can be replaced by an equivalent circuit (Fig. b) consisting of a voltage source V_{TH} in series with a resistor R_{TH},



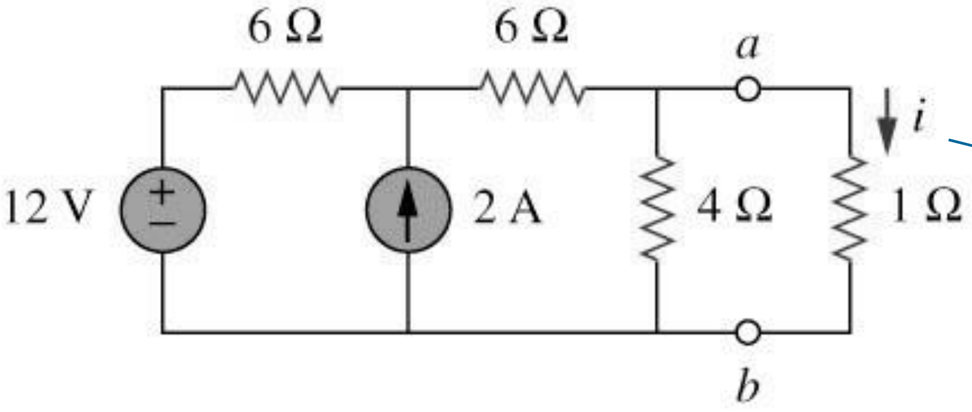
where

- ***VTH*** is the **open-circuit voltage** at the terminals.
- ***RTH*** is the input or equivalent resistance at the terminals when the **independent sources are turned off,** sometimes called the "Lookback Resistance"

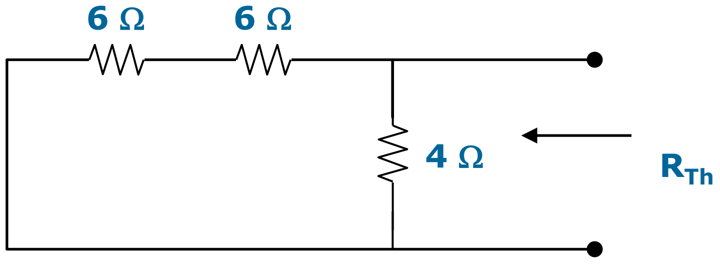
4.5 Thevenin's Theorem (2)

Example 5

Using Thevenin's theorem, find the equivalent circuit to the left of the terminals in the circuit shown below. Then find i.

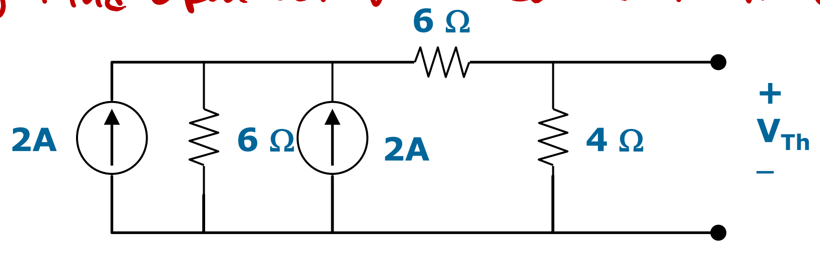


a) Turn off Indep Srcs to find R_{th}



(a)

b) Find open-cct V at desired terminals

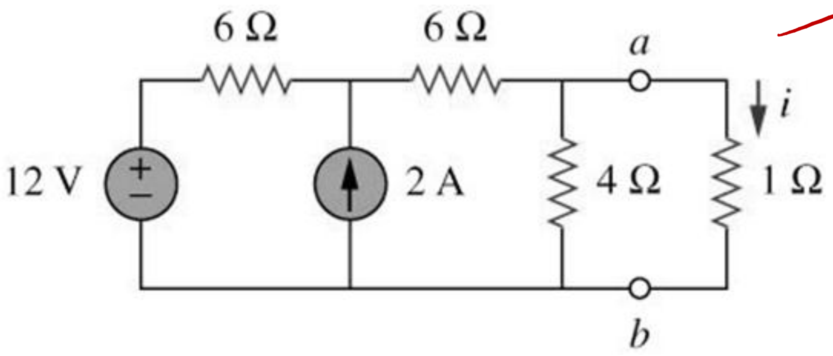


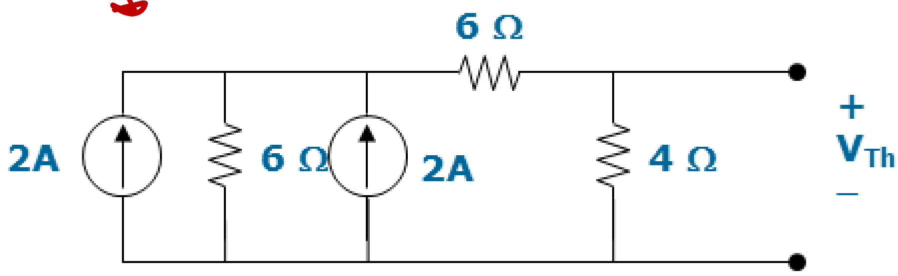
(b)

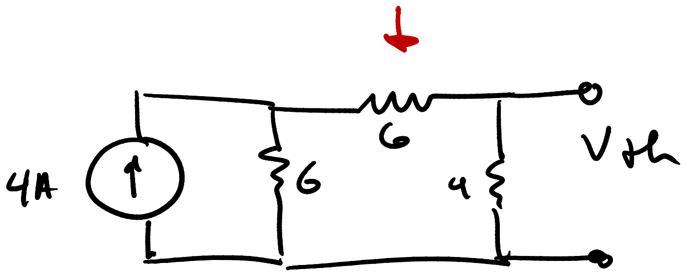
*Refer to in-class illustration, textbook, answer V_{TH} = 6V, R_{TH} = 3Ω, i = 1.5A

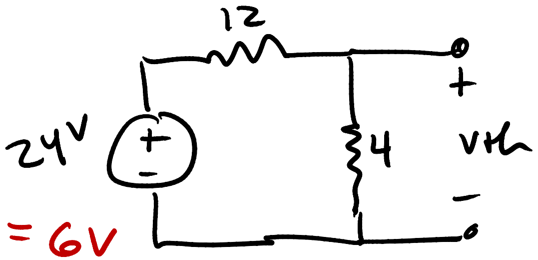
You can use any technique to find Vth

Nodal/Mesh Analysis, Source Xform, KCL/KVL

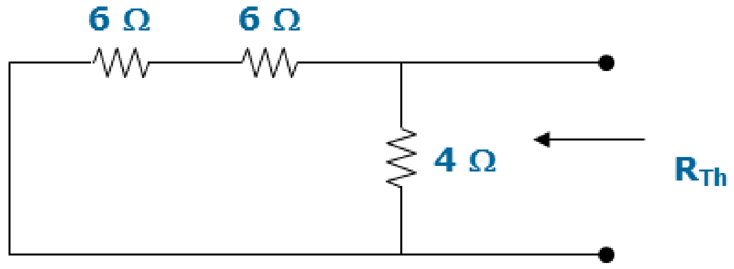






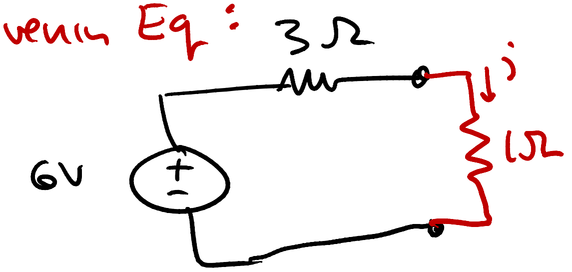


$V_{th} = 24\left(\frac{4}{16}\right) = 6V$



$R_{th} = 4 \| 12 = \frac{48}{16} = 3$

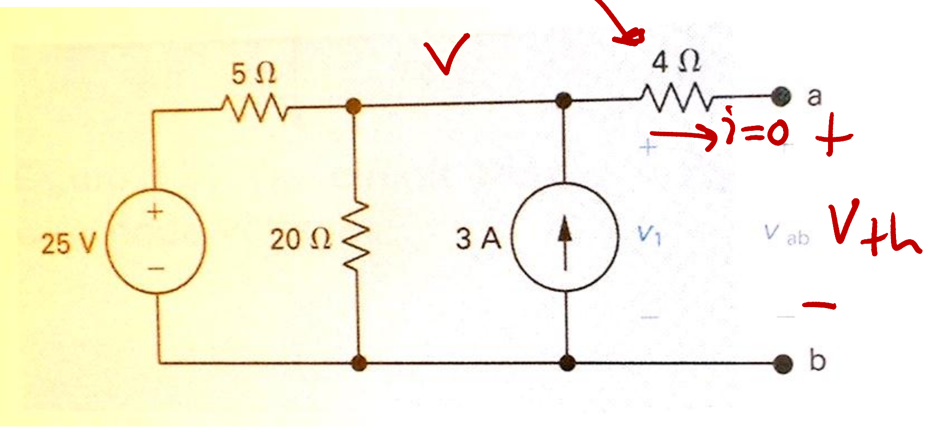
Thevenin Eq:



now $i = \frac{6}{4} = 1.5A$

3) Find Thevenin Eq at a-b

note how R4 does not affect Voc! [Vth=32, Rth=8]

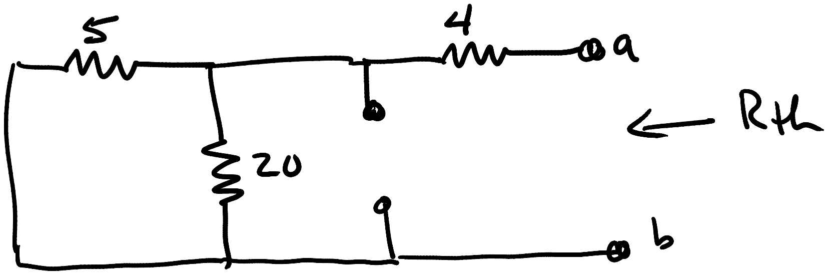


Let's use NV to find V = Vth

$$\left(\frac{V-25}{5}+\frac{V}{20}-3=0\right)\times 20 \rightarrow$$

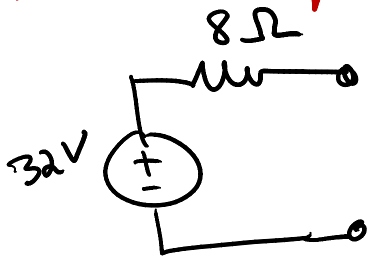
$$5V = 160\ ,\quad \underline{\underline{V_{th}=32}}$$

Turn off Sources to find Rth

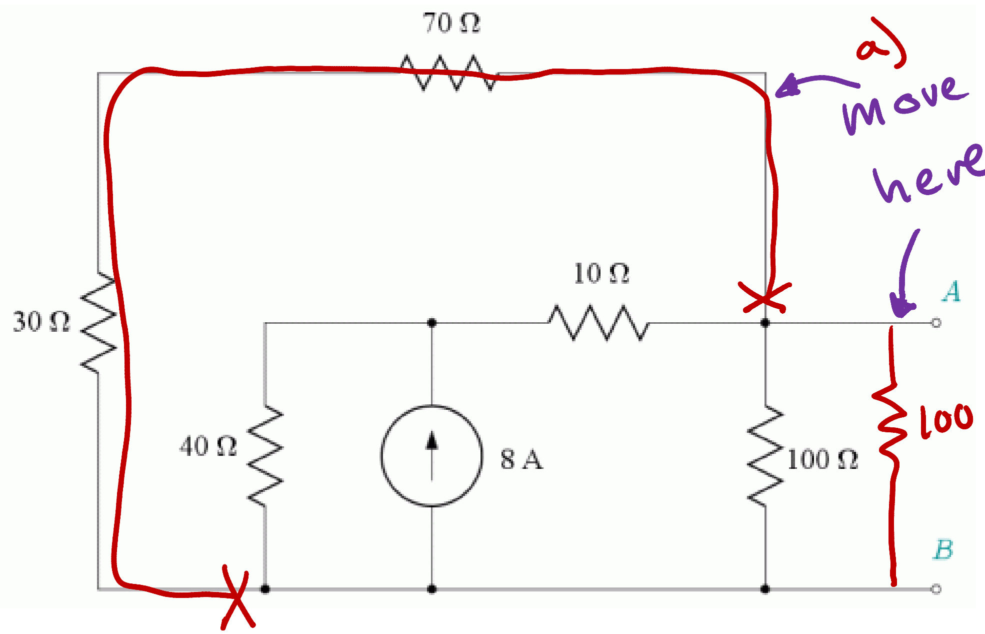


$$R_{th} = 4 + 5\|20 = \underline{\underline{8\,\Omega}}$$

Thevenin Eq:

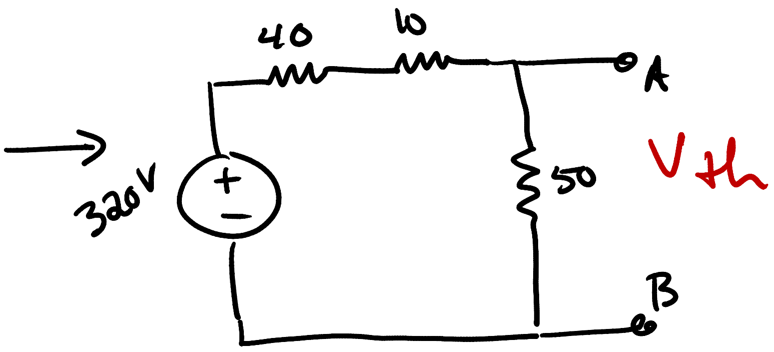


4) Find the Thèvenin equivalent circuit at the terminals A-B. Rt = 25Ω;Vt = 160V



a) move this branch here

b) Xform I → V src & combine || R's

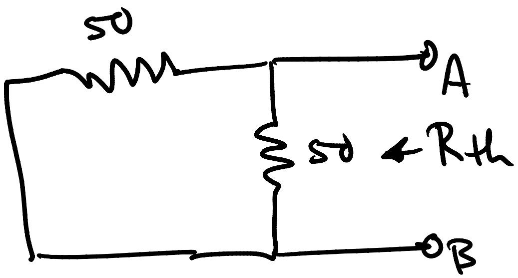


c) Use voltage divider formula

$$V_{th} = \left(\frac{50}{100}\right)320 = \underline{\underline{160V}}$$

$$R_{th} = 50 \| 50 = \underline{\underline{25\Omega}}$$

d) Turn off SRC for Rth



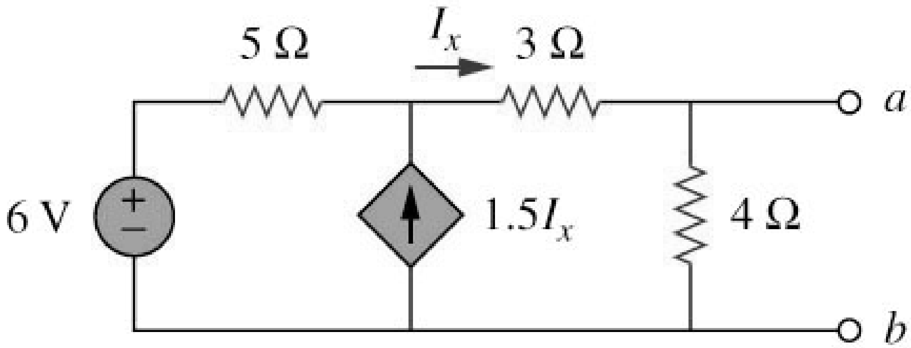
Finding Thevenin Eq with Dependent Sources

- Vth is still the open circuit voltage
  - across the terminals of interest
- To find Rth,
  - we CAN turn off Independent Sources
  - we CANNOT turn off Dependent Sources
    - they depend on other circuit variables
  - SOLN: hook up an external source at terminals, find Vo/Io = Rth

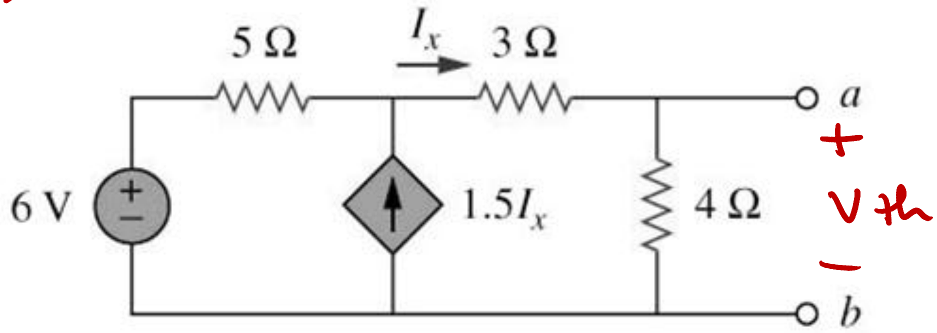
4.5 Thevenin's Theorem (3)

Example 6

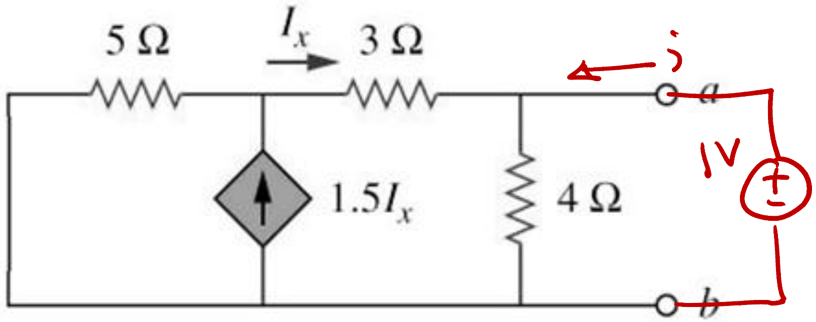
Find the Thevenin equivalent circuit of the circuit shown below to the left of the terminals.



a) Find Vth as before (Voc at a-b)

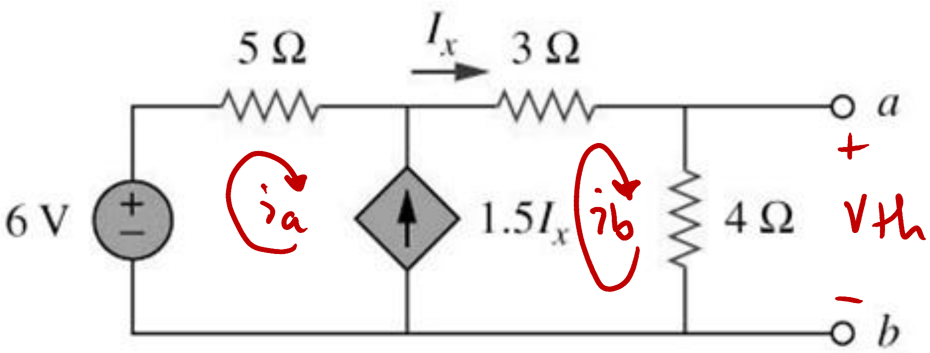


b) Find Rth by i) turn off indep src
ii) connect external src



*Refer to in-class illustration, textbook, answer V_{TH} = 5.33V, R_{TH} = 0,444Ω

Finding Vth using Mesh Analysis



Supermesh:

$-6 + 5i_a + 3i_b + 4i_b = 0$

Constraint Eq:

$i_b - i_a = 1.5 I_x = 1.5 i_b$

or, $-.5 i_b = i_a$

Also

$i_b = I_x$

Substituting $i_a = -.5 i_b$ into top eq

$-2.5 i_b + 7 i_b = 6$

$i_b = \frac{6}{4.5} = 1.333$

$V_{th} = 4 i_b = 5.333\ V$

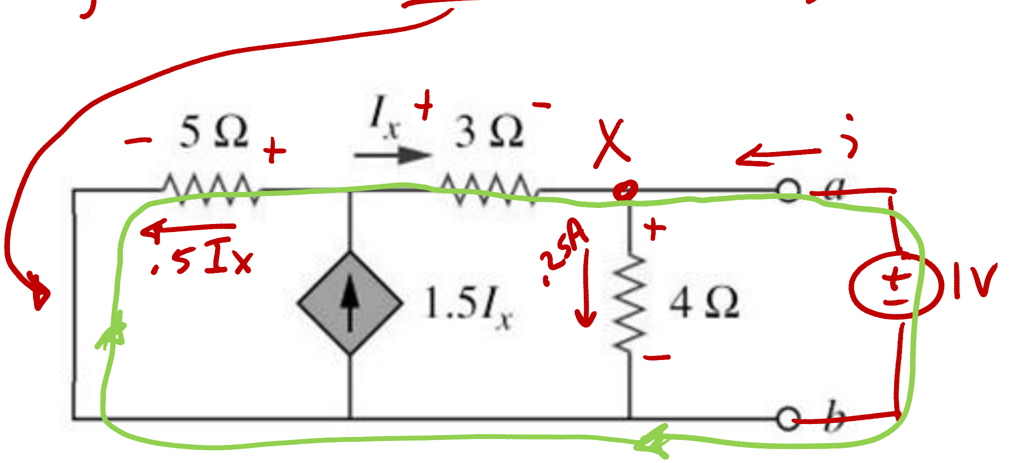
Finding Rth in cct with Depend Src

a) Turn off Indep Src

b) Connect External Src (1V, could be anything)

c) solve for i

d) $R_{th} = \frac{1V}{i}$



c) KCL at X

$I_x + i = .25A$

KVL Green Loop

$-5(.5I_x) + 3I_x + 1V = 0$

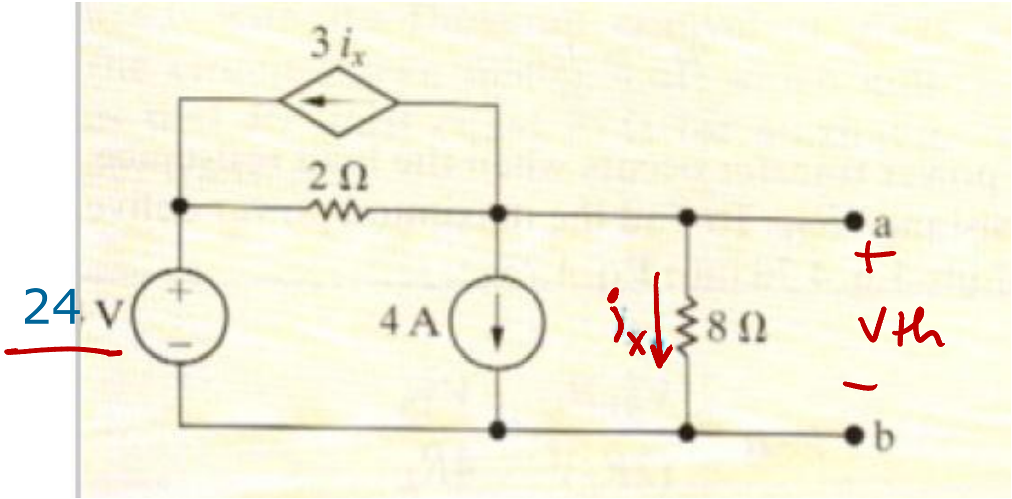
$.5I_x = -1$

$I_x = -2$

$\therefore\ i = .25A - I_x = 2.25A$

d) $R_{th} = \frac{1V}{2.25A} = \underline{.44\Omega}$

5) Find Thevenin at a-b (Vth=8, Rth=1)



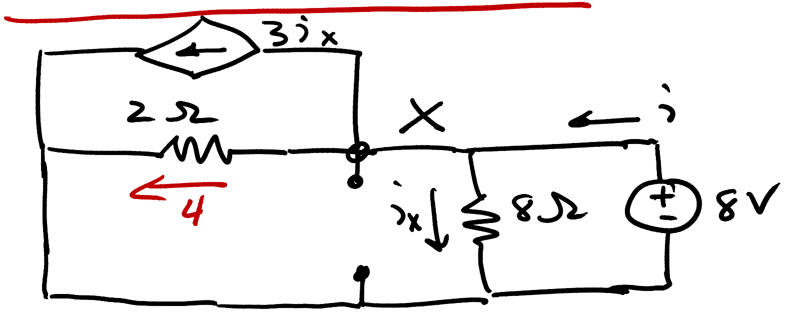
Nodal Analysis for V th

$$\frac{V_{th}}{8} + 4 + \frac{V_{th}-24}{2} + 3i_x = 0$$

$$i_x = \frac{V_{th}}{8}$$

$$8V_{th} = 64 \longrightarrow V_{th} = \underline{\underline{8V}}$$

External src for Rth



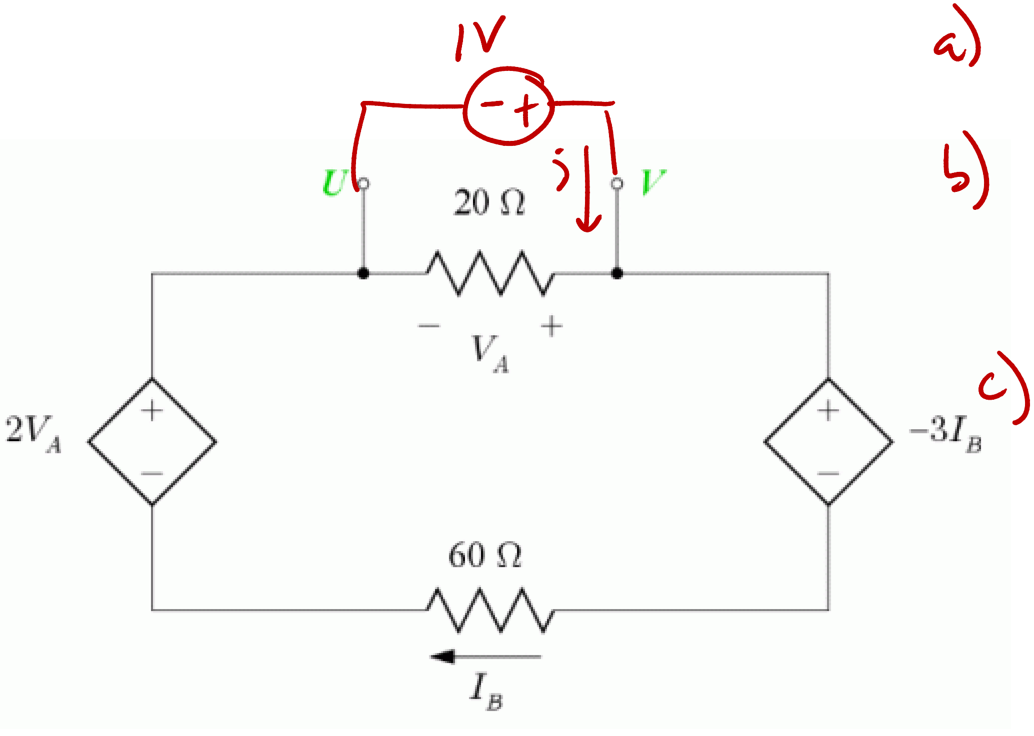
OL

$i_x = 1\,A$

KCL at Node X

$$i = 1 + 4 + 3 = 8A \longrightarrow R_{th} = \frac{8V}{8A} = \underline{\underline{1\,\Omega}}$$

6) Find the Thèvenin equivalent circuit at the terminals U-V. (Vth=0, Rth = 9.74)



a) No Indep Src → $V_{th} = 0$

b) Connect 1V Source

$V_A = 1V$

c) KVL

$-2V_a - 1V + (-3I_B) + 60 I_B = 0$

$57 I_B = 3V$

$I_B = \frac{3}{57} A$

d) KCL at node V

$i = \frac{1}{20} A + I_B = \frac{1}{20} + \frac{3}{57} = .1026A$

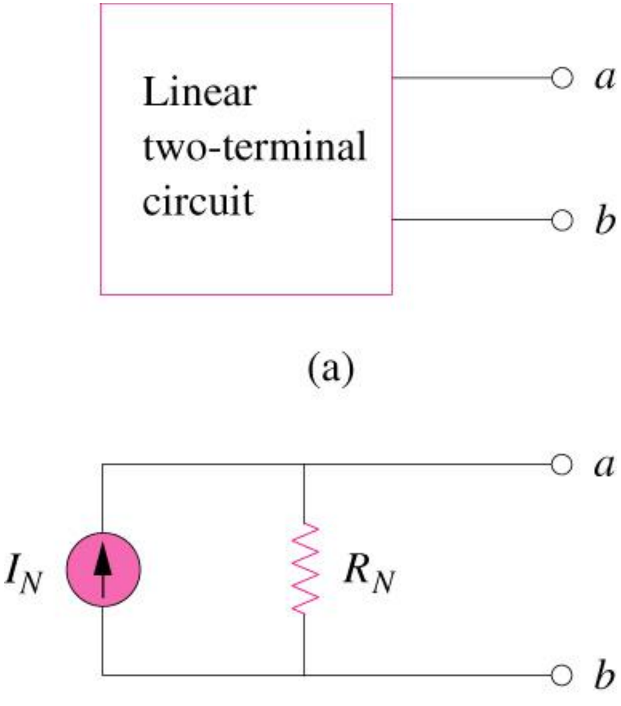
e) $R_{th} = \frac{1V}{.1026A} = \underline{\underline{9.74\ \Omega}}$

4.6 Norton’s Theorem (1)

It states that a linear two-terminal circuit can be replaced by an equivalent circuit of a current source I_N in parallel with a resistor R_N,

Where

- I_N is the short circuit current through the terminals.
- R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.



(a)

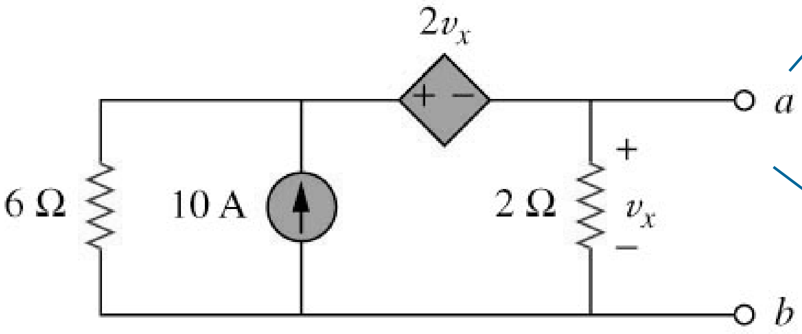
(b)

The Thevenin’s and Norton equivalent circuits are related by a source transformation.

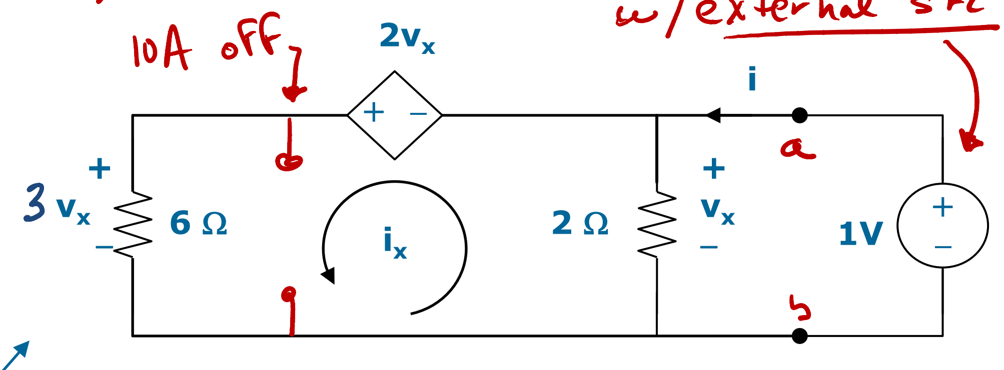
4.6 Norton's Theorem (2)

Example 7

Find the Norton equivalent circuit of the circuit shown below.

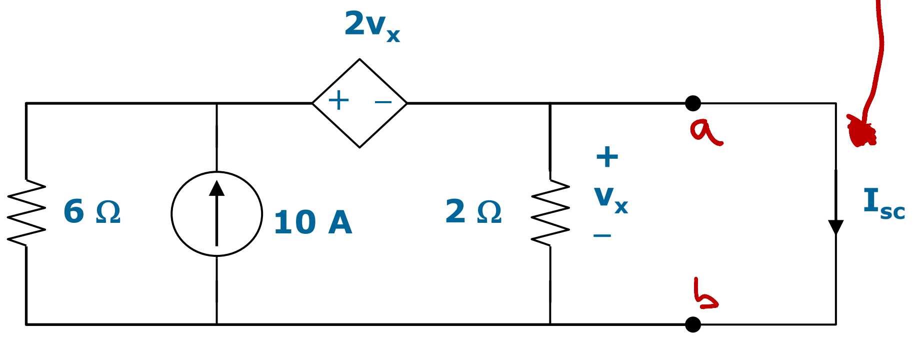


a) Find $R_N = R_{th}$



(a)

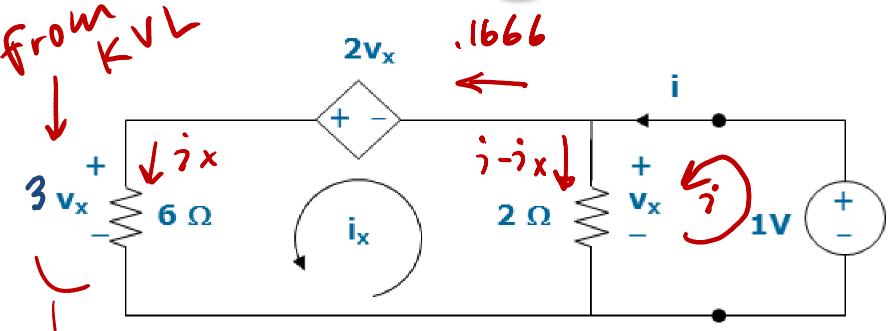
b) Find $I_N = I_{sc}$, short circuit current thru a-b

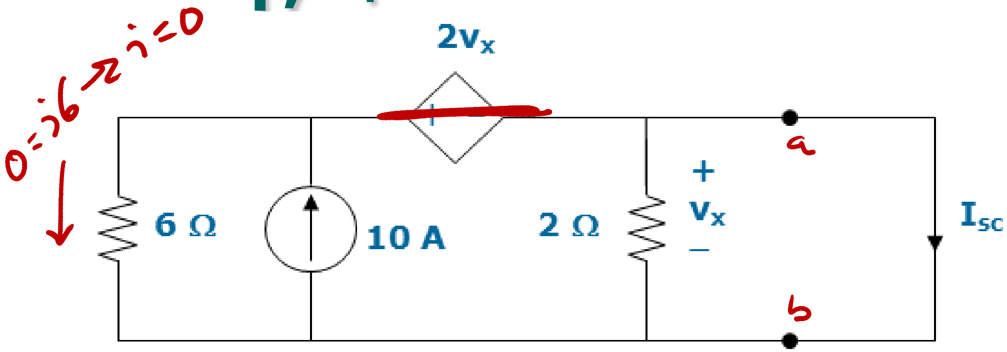


(b)

*Refer to in-class illustration, textbook, R_N = 1Ω, I_N = 10A.

Finding Norton Eq, previous slide





$V_x = 1V$

$i_x = \frac{3V_x}{6} = .5A$

$2(i - i_x) = 1V$

$i = .5 + i_x = 1A$

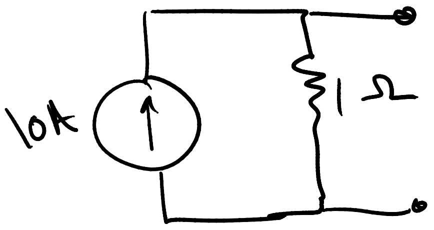
$R_N = \frac{1V}{1A} = \underline{\underline{1\Omega}}$

Shorting a-b sets $V_x = 0$
& $2V_x = 0$

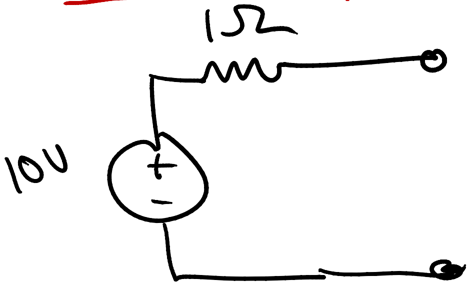
All current goes thru a-b
path of least resistance

$I_{sc} = 10A = \underline{\underline{I_N}}$

NORTON EQUIV



Thevenin Equiv



- Or find Vth and calculate Isc = Vth/Rth

4.7 Maximum Power Transfer (1)

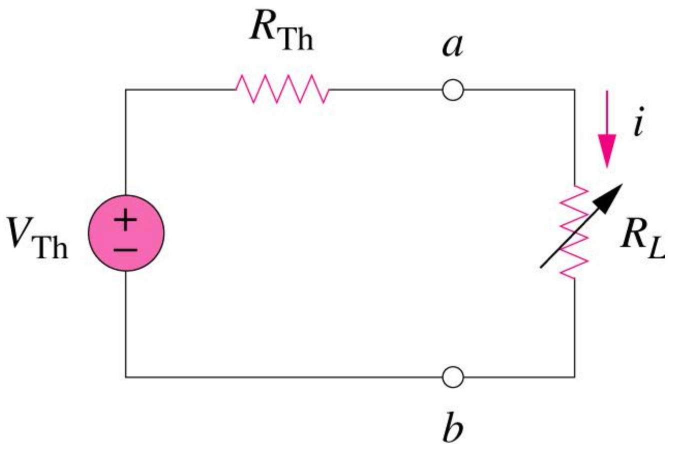
If the entire circuit is replaced by its Thevenin equivalent except for the load, the power delivered to the load is:

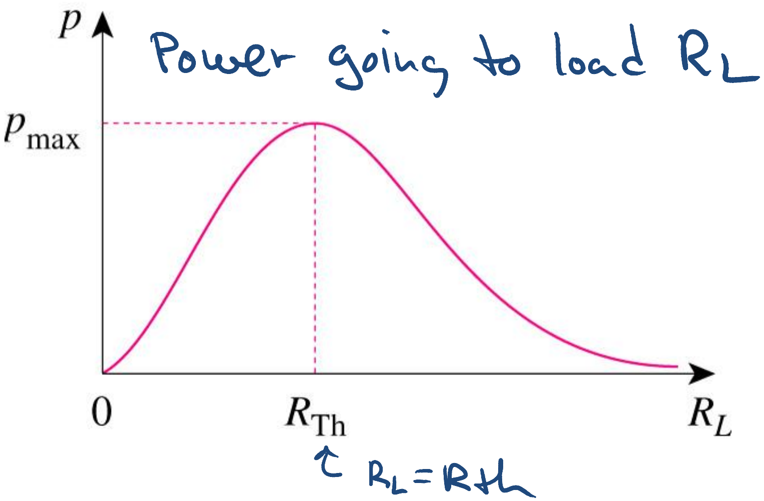
$$P = i^2 R_L = \left(\frac{V_{th}}{R_{th} + R_L} \right)^2 R_L$$

To max power going to R_L,

set $R_L = R_{th}$

then $P_{max} = \frac{V_{th}^2}{4 R_L}$





Power going to load R_L

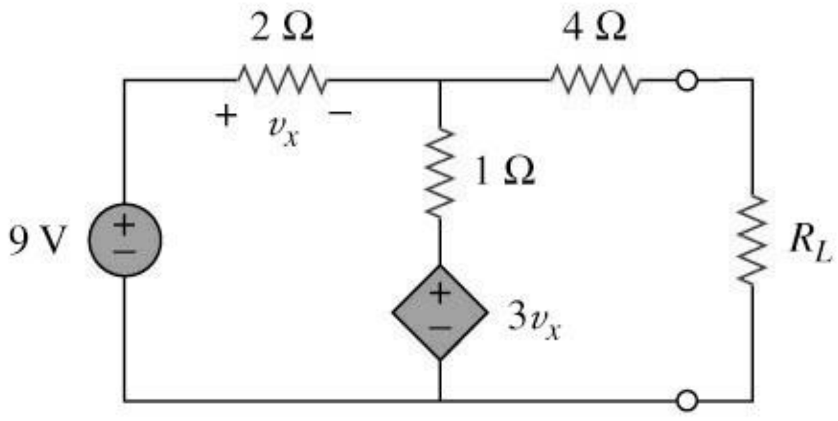
↑ $R_L = R_{th}$

The power transfer profile with different R_L

4.7 Maximum Power Transfer (2)

Example 8

Determine the value of R_L that will draw the maximum power from the rest of the circuit shown below. Calculate the maximum power.



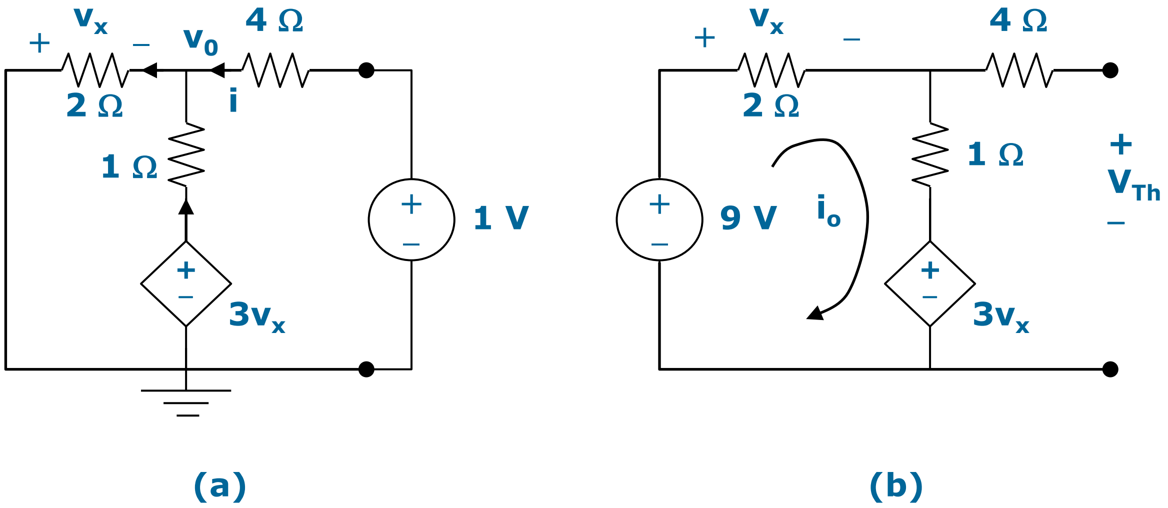


Fig. a

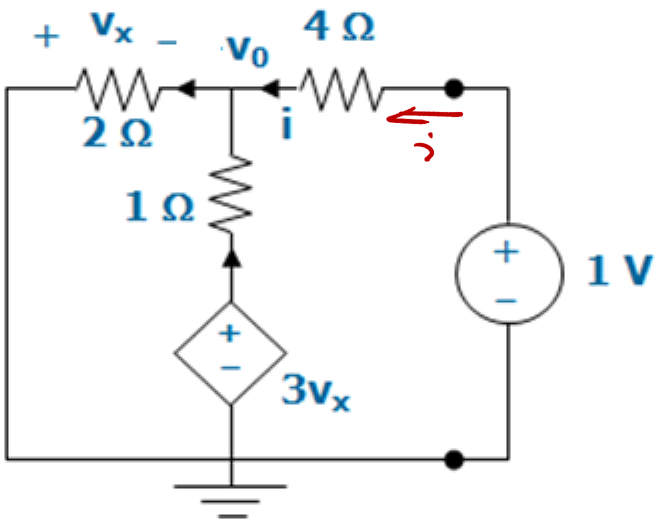
=> To determine R_{TH}

Fig. b

=> To determine V_{TH}

*Refer to in-class illustration, textbook, $R_L = 4.22\Omega$, $P_m = 2.901W$

a) Find Rth



Use Nodal Analysis

$$\frac{v_0}{2} + \frac{v_0 - 3v_x}{1} + \frac{v_0 - 1}{4} = 0$$

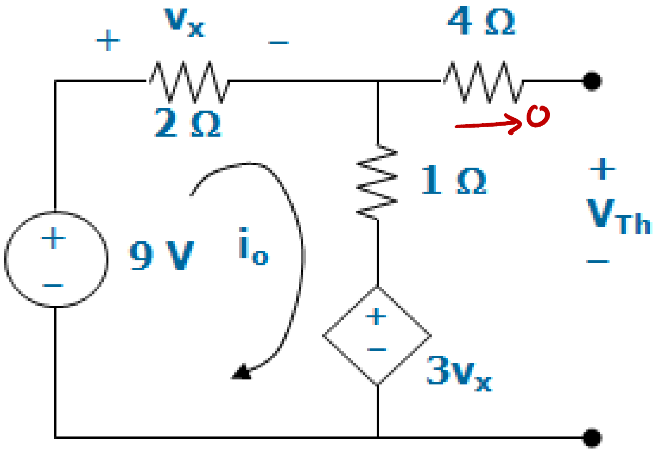
$$v_x = -v_0$$

$$19 v_0 = 1 , \quad v_0 = \frac{1}{19} = .0526 \text{ V}$$

$$i = \frac{1 - v_0}{4} = \frac{1 - .0526}{4} = .2368$$

$$R_{th} = \frac{1}{.2368} = \underline{\underline{4.22\ \Omega}}$$

b) Find Vth



Use Mesh Analysis

$$-9 + 2i_o + 1i_o + 3v_x = 0$$

$$v_x = 2i_o$$

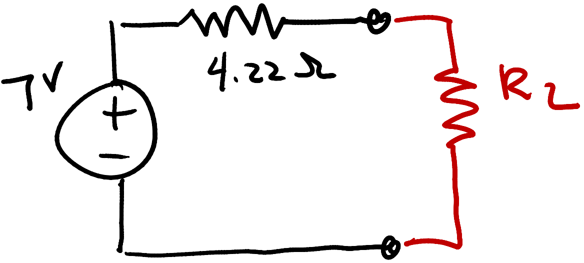
$$9i_o = 9 \; , \quad i_o = 1A$$

KVL

$$-3V_x - 1i_o + 4(0) + V_{th} = 0$$

$$V_{th} = 3V_x + 1i_o = \underline{\underline{7V}}$$

Thevenin Eq Cct:



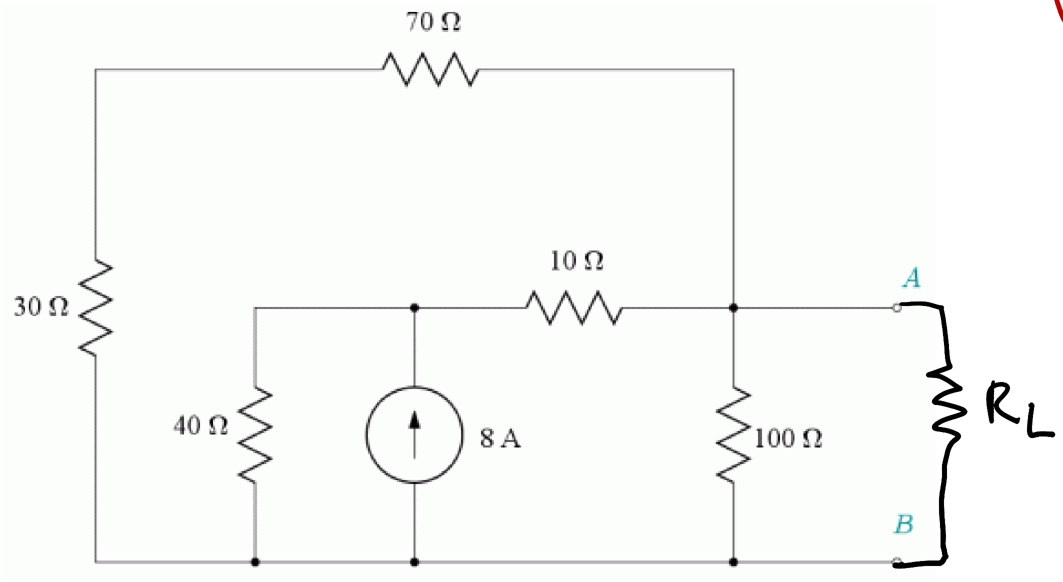
$$R_L = R_{th} = \underline{4.22\Omega}$$

$$P_{max} = \frac{V_{th}^2}{4R_{th}} = \frac{7^2}{4(4.22)} = \underline{2.901W}$$

c) Set RL = Rth

Pmax = $V_{th}^2/(4R_{th})$ [2.901W]

7) What is the most power we can deliver to RL? (use Vth, Rth from prob 4) [256 W]



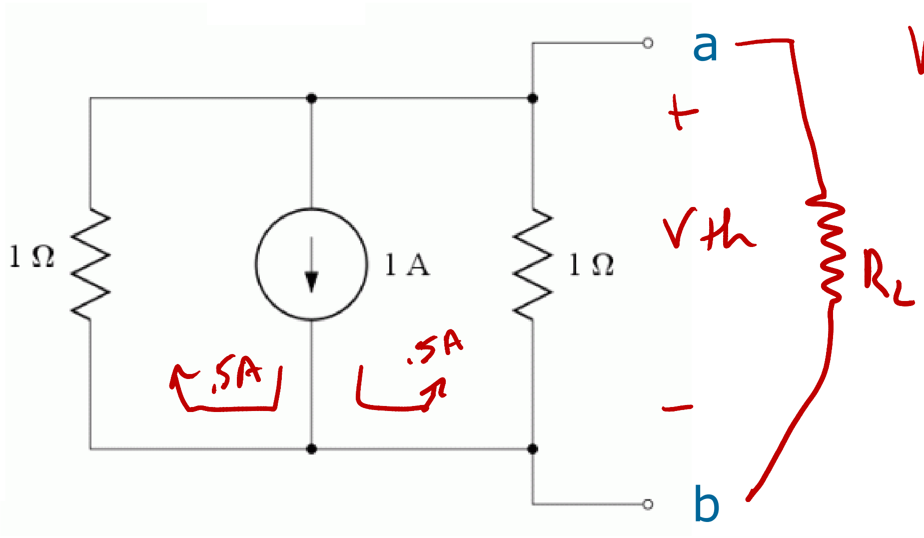
$V_{th} = 160 V$

$R_{th} = 25 \Omega$

Set $R_L = R_{th} = 25 \Omega$

$P_{max} = \frac{160^2}{4(25)} = 256 W$

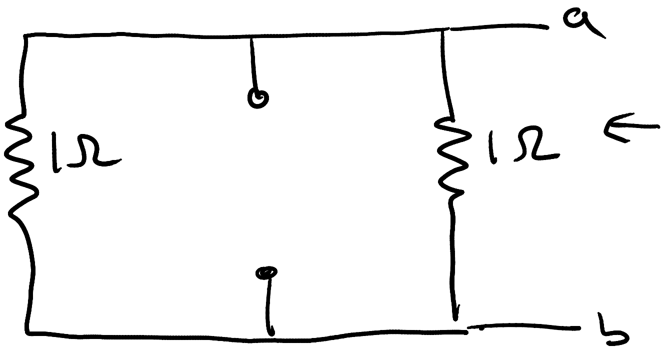
8) Find the Max Power that can be delivered to a load at a-b [125 mW]



$V_{th} = -.5V$

← Set $R_L = R_{th} = 0.5\Omega$

$$P_{max} = \frac{(-.5)^2}{4(.5)} = \underline{\underline{.125W}}$$



$R_{th} = 1||1 = .5\Omega$

Summary of Thevenin/Norton

- Vth = open cct voltage at terminals
- Rth = equivalent Resistance seen at terminals when all independent sources turned off
- In = current through terminals when they are shorted
- Rn = Rth
- In = Vth/Rth, Vth = In * Rn

Finding $R_{thevenin}$

- IF Circuit has ONLY Independent Srcs:
  - Option A:
    - Turn off all sources
    - Find equivalent resistance between terminals
  - Option B:
    - Find Vth, In (short circuit current)
    - Rth = Vth/In

Finding $R_{thevenin}$

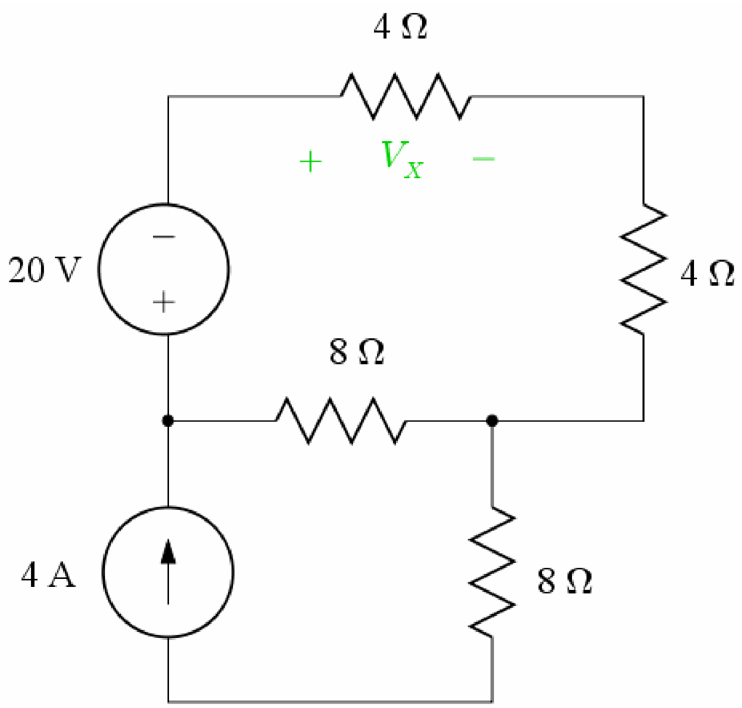
- If Circuit has Dependent + Independnt Sources:
  - Option A:
    - Turn off all INDEP sources
    - Connect an External Source to terminals (Vo or Io)
    - Find the other quantity (Io or Vo)
    - Rth = Vo/Io
  - Option B:
    - Find Vth (open cct volts), In (short cct current)
    - Rth = Vth/In

Finding Rthevenin

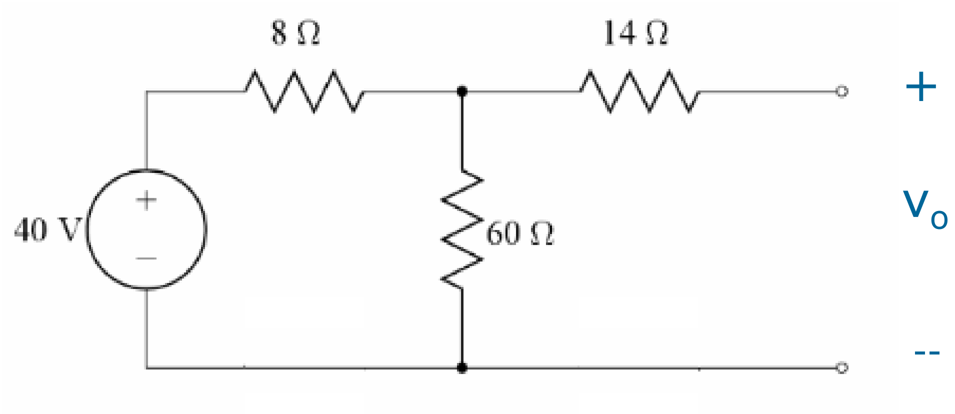
- Circuit has ONLY Dependent Sources:
  - Only Option:
    - Connect an External Source to terminals (Vo or Io)
    - Find the other quantity (Io or Vo)
    - Rth = Vo/Io

Handouts

1) Use Superposition to find Vx (3V)

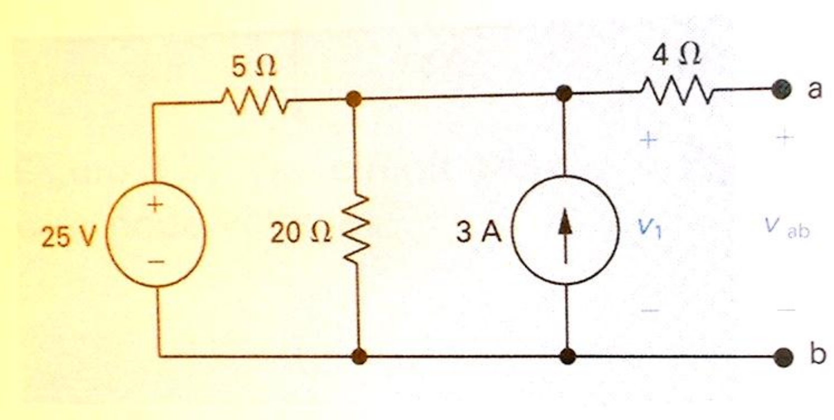


2) Find v_o in the circuit shown below using source transformation.

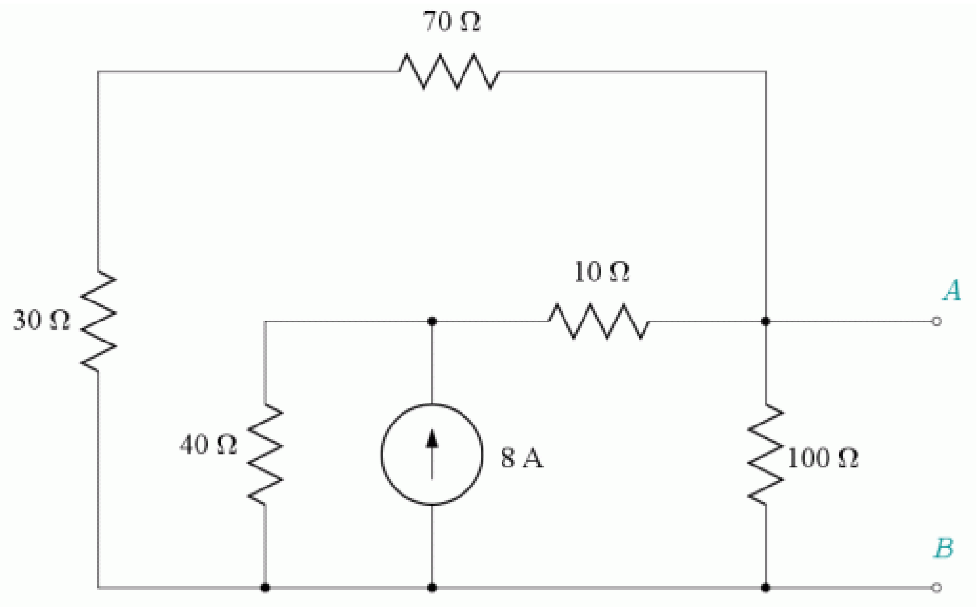


3) Find Thevenin Eq at a-b

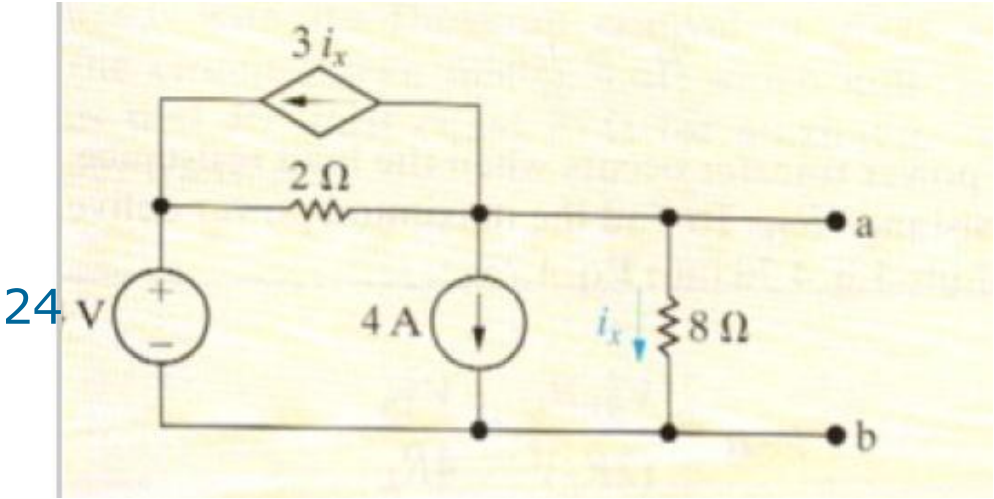
note how R4 does not affect Voc! [Vth=32, Rth=8]



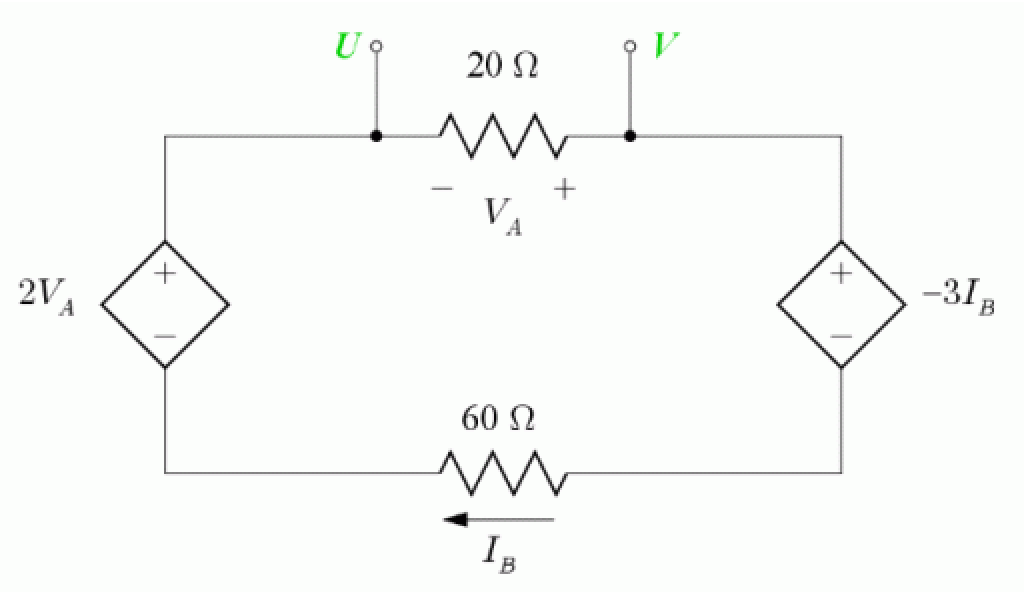
4) Find the Thèvenin equivalent circuit at the terminals A-B. Rt = 25Ω;Vt = 160V



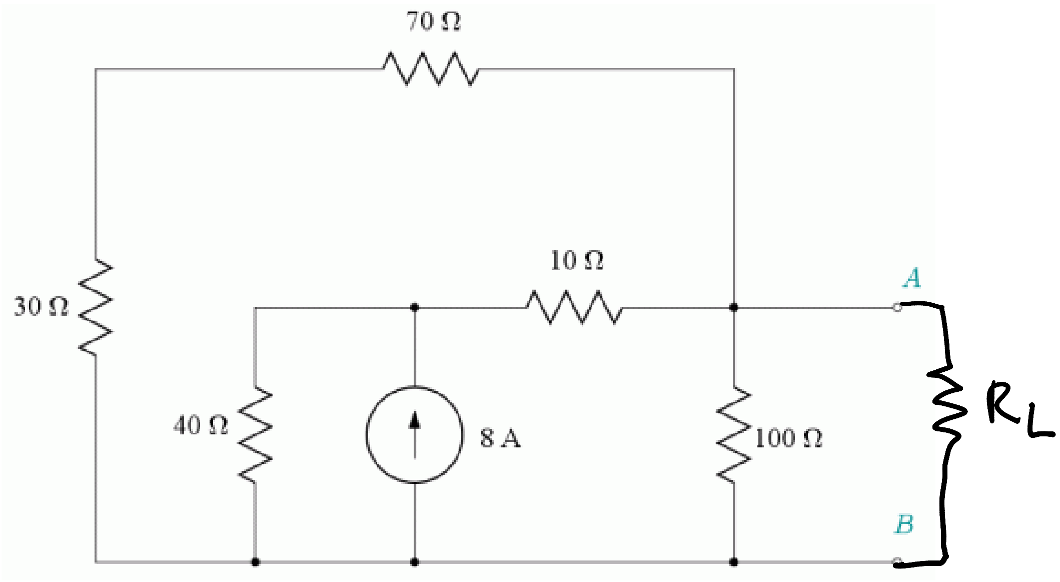
5) Find Thevenin at a-b (Vth=8, Rth=1)



6) Find the Thèvenin equivalent circuit at the terminals U-V. (Vth=0, Rth = 9.74)



7) What is the most power we can deliver to RL? (use Vth, Rth from prob 4) [256 W]



8) Find the Max Power that can be delivered to a load at a-b [125 mW]