Circuit Theory

Chapter 14

Frequency Response

Frequency Response
Chapter 14

14.1 Introduction

14.2 Transfer Function

14.3 Series Resonance

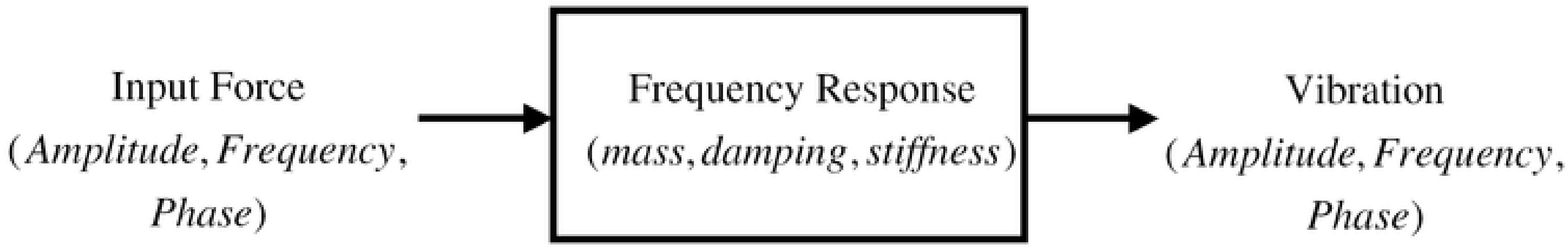
14.4 Parallel Resonance

14.5 Passive Filters

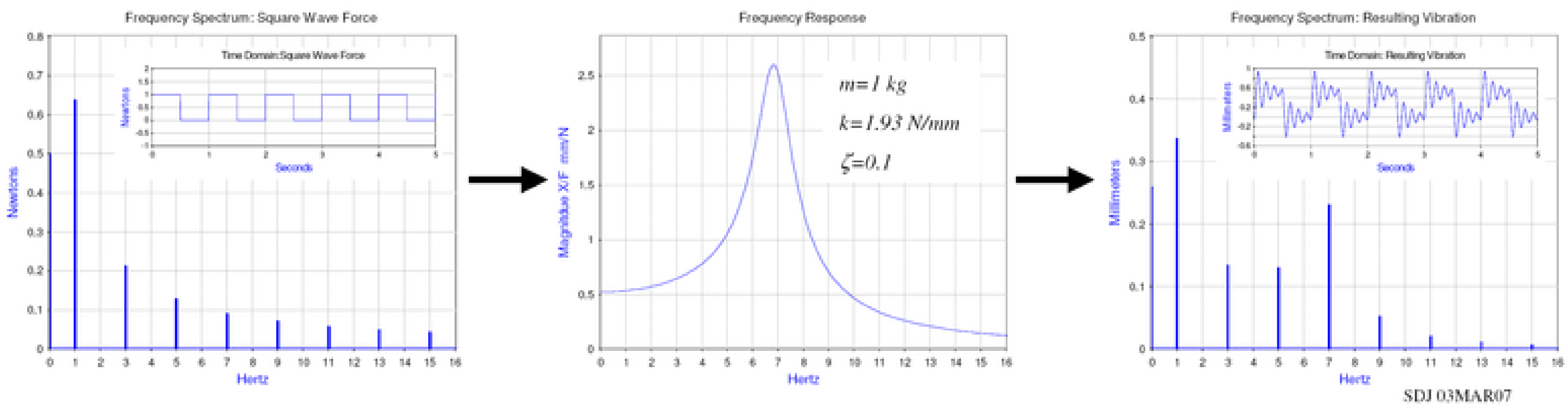
14.1 Introduction (1)

What is FrequencyResponse of a Circuit?

It is the variation in a circuit's behavior with change in signal frequency and may also be considered as the variation of the gain and phase with frequency.



$$F(\omega) \quad x \quad H(\omega) \quad = \quad X(\omega)$$

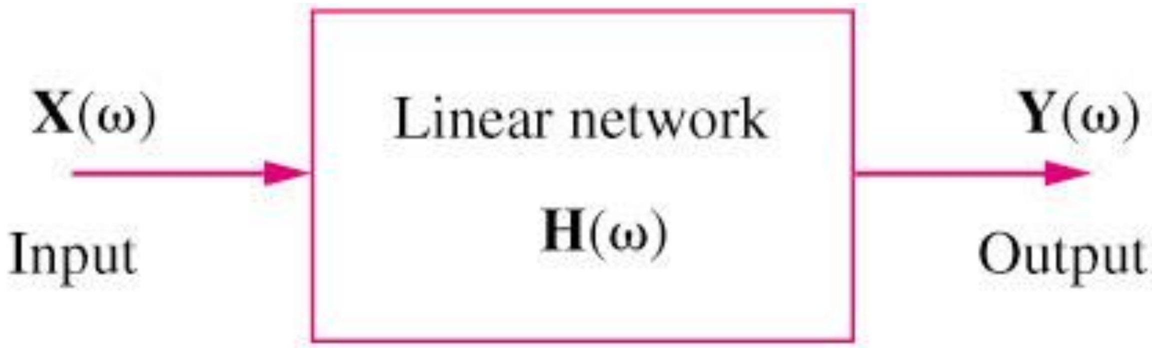


http://en.wikipedia.org/wiki/Tacoma_Narrows_Bridge_(1940)

http://lpsa.swarthmore.edu/Analogs/ElectricalMechanicalAnalogs.html

14.2 Transfer Function (1)

- The transfer function H(ω) of a circuit is the **frequency-dependent ratio** of a phasor output Y(ω) (an element voltage or current) to a phasor input X(ω) (source voltage or current).



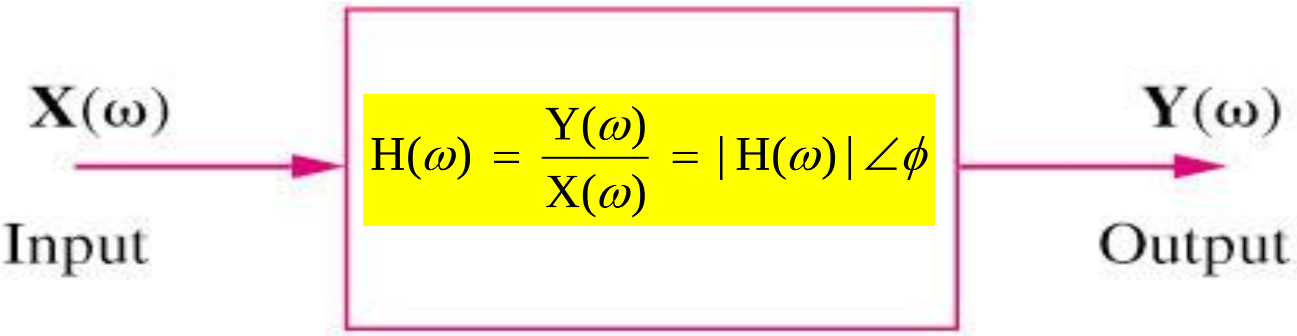
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = |H(\omega)| \angle \phi$$

14.2 Transfer Function (2)

- Four possible transfer functions:

$$H(\omega) = \text{Voltage gain} = \frac{V_o(\omega)}{V_i(\omega)}$$

$$H(\omega) = \text{Transfer Impedance} = \frac{V_o(\omega)}{I_i(\omega)}$$



$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = |H(\omega)| \angle \phi$$

$$H(\omega) = \text{Current gain} = \frac{I_o(\omega)}{I_i(\omega)}$$

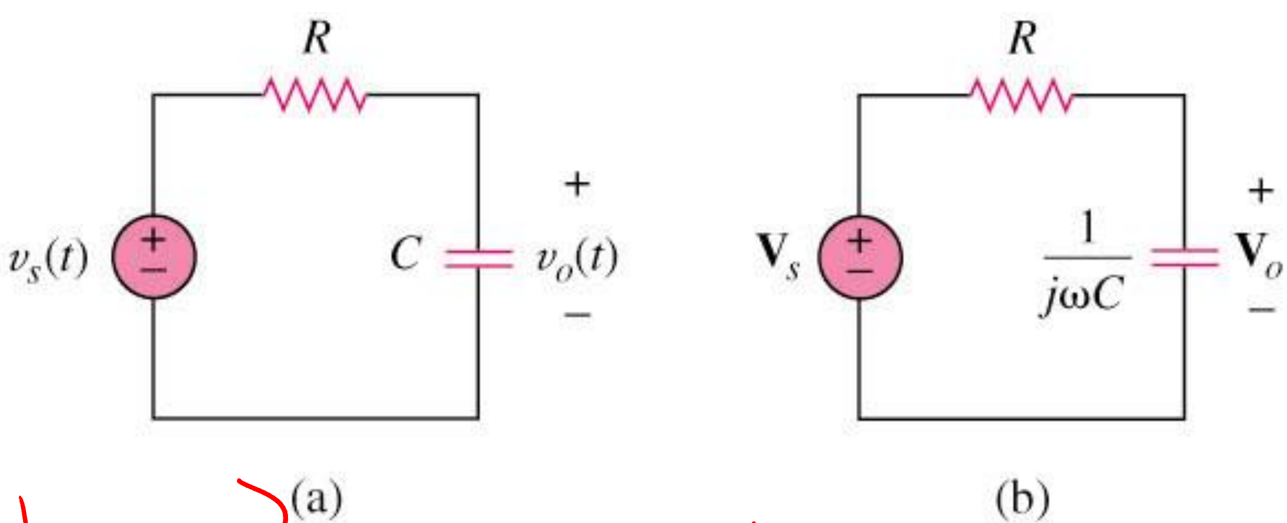
$$H(\omega) = \text{Transfer Admittance} = \frac{I_o(\omega)}{V_i(\omega)}$$

Example 1

For the RC circuit shown below, obtain the transfer function Vo/Vs and its frequency response. Let $v_s = V_m \cos\omega t$.

Voltage Divider

$$V_o = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} V_s$$



$$H(\omega) = \frac{V_o}{V_s} = \frac{1}{1 + j\omega RC} \rightarrow H(\omega) = \frac{1}{1 + j\frac{\omega}{\omega_0}}$$

let $\omega_0 = \frac{1}{RC}$

Polar form

$$H(\omega) = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \angle \tan^{-1}\left(\frac{\omega}{\omega_0}\right)} = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}} \angle -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

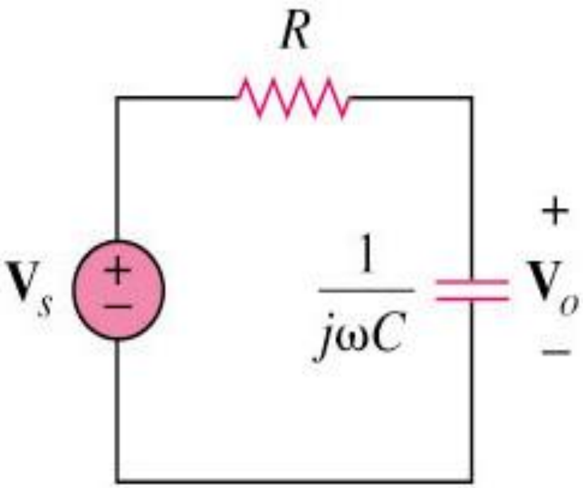
magnitude — Phase angle

$\omega = 0$
$H(0) = 1\angle 0^\circ$

$\omega = \infty$
$H(\infty) = 0\angle -90^\circ$

$\omega = \omega_0$
$H(\omega) = .707\angle -45^\circ$

Low Pass Filter



Solution:

The transfer function is

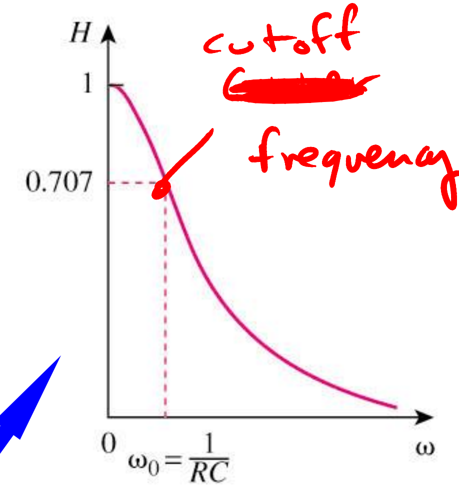
$$H(\omega)=\frac{V_o}{V_s}=\frac{\frac{1}{j\omega C}}{R+1/j\omega C}=\frac{1}{1+j\omega RC}$$

The magnitude is $$|H(\omega)|=\frac{1}{\sqrt{1+(\omega/\omega_o)^2}}$$

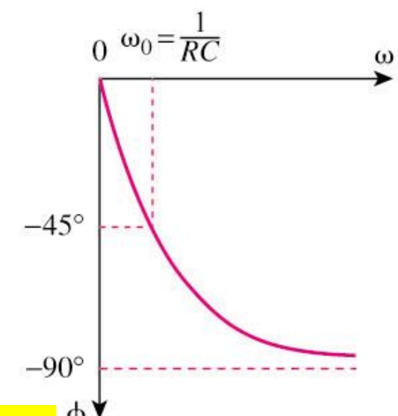
The phase is $$\phi=-\tan^{-1}\frac{\omega}{\omega_o}$$

$$\omega_o=1/RC$$

Low Pass Filter

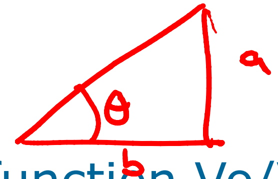


(a)



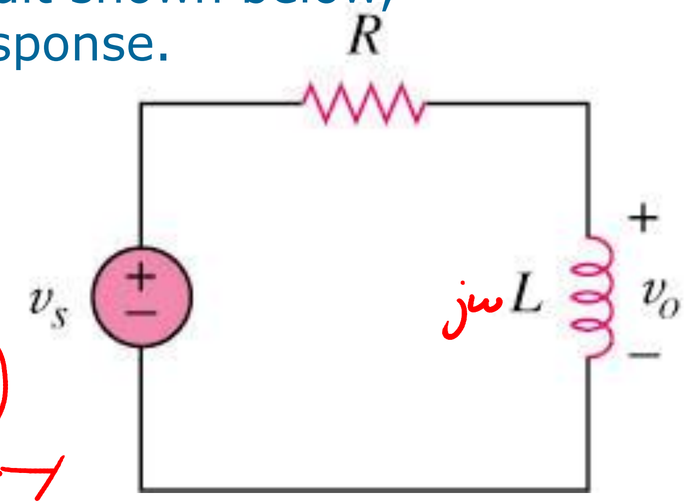
(b)

Example 2



$\tan^{-1}\left(\frac{a}{b}\right) = 90 - \tan^{-1}\left(\frac{b}{a}\right)$

Obtain the transfer function Vo/Vs of the RL circuit shown below, assuming $v_s = V_m\cos\omega t$. Sketch its frequency response.



$$\mathbf{V}_o = \frac{j\omega L}{R + j\omega L}\mathbf{V}_s$$

$$H(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{j\omega L / j\omega L}{(R + j\omega L)/j\omega L} = \frac{1}{1 + \frac{R}{j\omega L}}$$

let $\omega_0 = \frac{R}{L}$

$$H(\omega) = \frac{1}{1 - j\frac{\omega_0}{\omega}} \xrightarrow[\text{Form}]{\text{Polar}} = \frac{1}{\sqrt{1 + \left(\frac{\omega_0}{\omega}\right)^2}\angle -\tan^{-1}\left(\frac{\omega_0}{\omega}\right)} = \frac{1}{\sqrt{1 + \left(\frac{\omega_0}{\omega}\right)^2}}\angle \tan^{-1}\left(\frac{\omega_0}{\omega}\right)$$

$$= \left(\frac{1}{\sqrt{1 + \left(\frac{\omega_0}{\omega}\right)^2}}\right)\angle 90 - \tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

$H(0) = 0\angle 90$

$H(\omega_0) = .707\angle 45^{\circ}$

$H(\infty) = 1\angle 0$

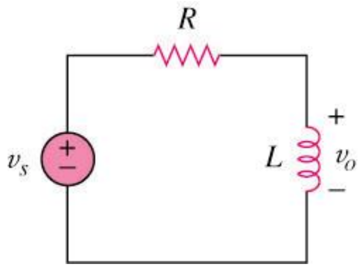
HIGH PASS FILTER

14.2 Transfer Function (6)

Solution:

The transfer function is

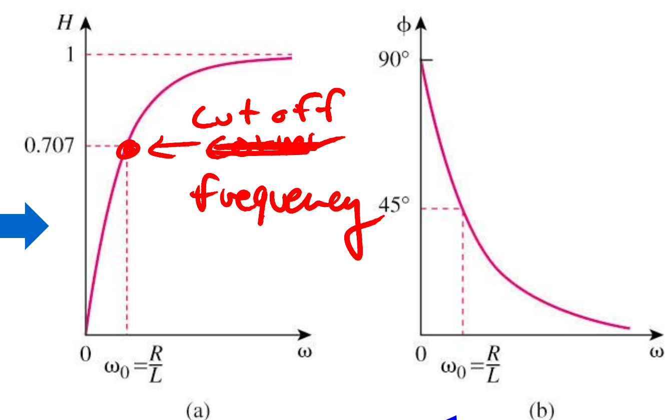
$$H(\omega) = \frac{V_o}{V_s} = \frac{j\omega L}{R + j\omega L} = \frac{1}{1 + \dfrac{R}{j\omega L}}$$



High Pass Filter

The magnitude is

$$H(\omega) = \frac{1}{\sqrt{1 + (\frac{\omega_o}{\omega})^2}}$$



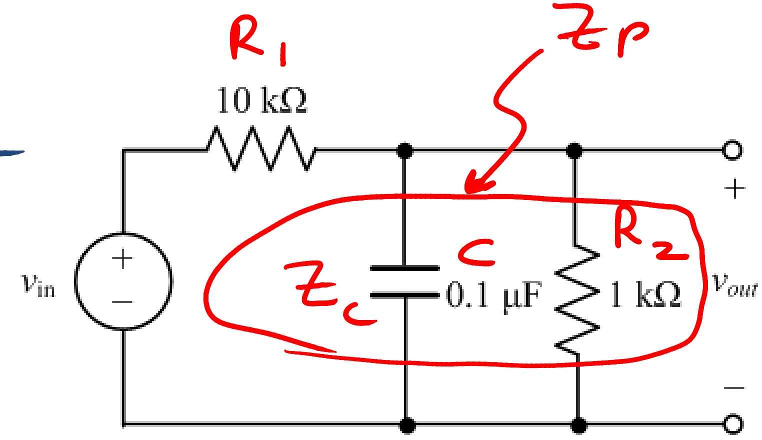
The phase is

$$\phi = \angle 90° - \tan^{-1}\frac{\omega}{\omega_o}$$

$$\omega_o = R/L$$

Ex3) Find the transfer function $H(\omega)=V_{out}/V_{in}$. What type of filter is it?

$$Z_c = \frac{1}{j\omega C}, \quad Z_p = \frac{R_2\left(\frac{1}{j\omega C}\right) j\omega C}{\left(R_2 + \frac{1}{j\omega C}\right) j\omega C}$$



$$Z_p = \frac{R_2}{1 + j\omega R_2 C}$$

$$V_{out} = \frac{V_{in} Z_p}{R_1 + Z_p} \Rightarrow H(\omega) = \frac{Z_p}{R_1 + Z_p} = \frac{\frac{R_2}{1 + j\omega R_2 C}(1 + j\omega R_2 C)}{\left(R_1 + \frac{R_2}{1 + j\omega R_2 C}\right)(1 + j\omega R_2 C)}$$

$$H(\omega) = \frac{R_2}{R_1 + j\omega R_1 R_2 C + R_2} = \frac{1000}{11000 + j\omega\, 10^4 10^3 10^{-7}} = \frac{1000}{11{,}000 + j\omega}$$

H(w)=1000/(jw+11000), Lowpass filter

Wolfram Alpha, add abs( ) for mag

www.wolframalpha.com/input/?i=plot+abs%281000%2F%28iw%2B11000%29%29%2C+w%2C+0%2C100000

plot abs(1000/(iw+11000)), w, 0,100000

Examples Random

Assuming "0" is referring to math | Use as a decimal number instead

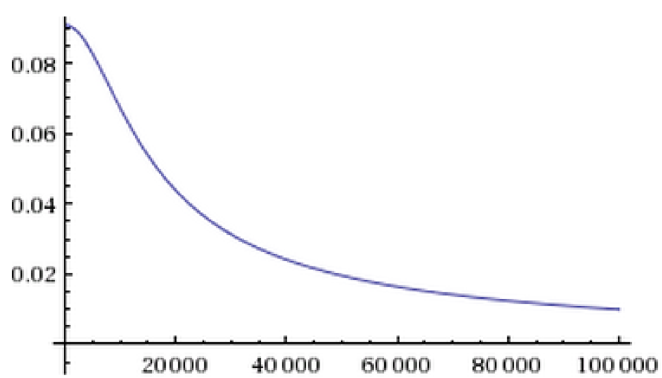
Assuming i is the imaginary unit | Use i as a variable instead

Input interpretation:

plot	$\frac{1000}{\lvert 11\,000 + i\,w \rvert}$	$w = 0$ to $100\,000$

$|z|$ is the absolute value of z »
i is the imaginary unit »

Plot:



Enable interactivity

Plotting phase in Wolfram

www.wolframalpha.com/input/?i=plot+180%2Fpi*arg%281000%2F%28iw%2B11000%29%29%2C+w%2C+0%2C100

plot 180/pi*arg(1000/(iw+11000)), w, 0,100000

Examples Random

Assuming "0" is referring to math | Use as a decimal number instead

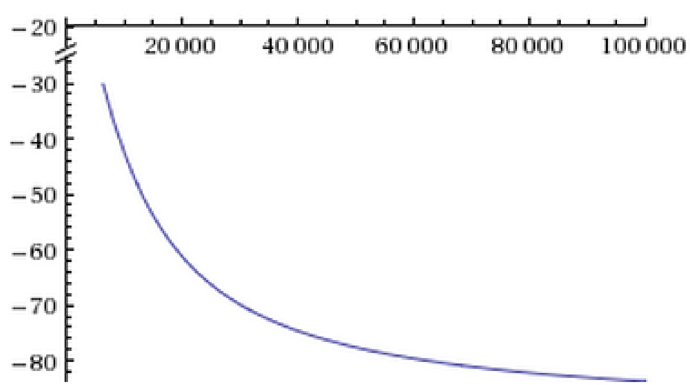
Assuming i is the imaginary unit | Use i as a variable instead

Input interpretation:

plot	$\frac{180 \arg\left(\frac{1}{11000 + i w}\right)}{\pi}$	$w = 0$ to 100 000

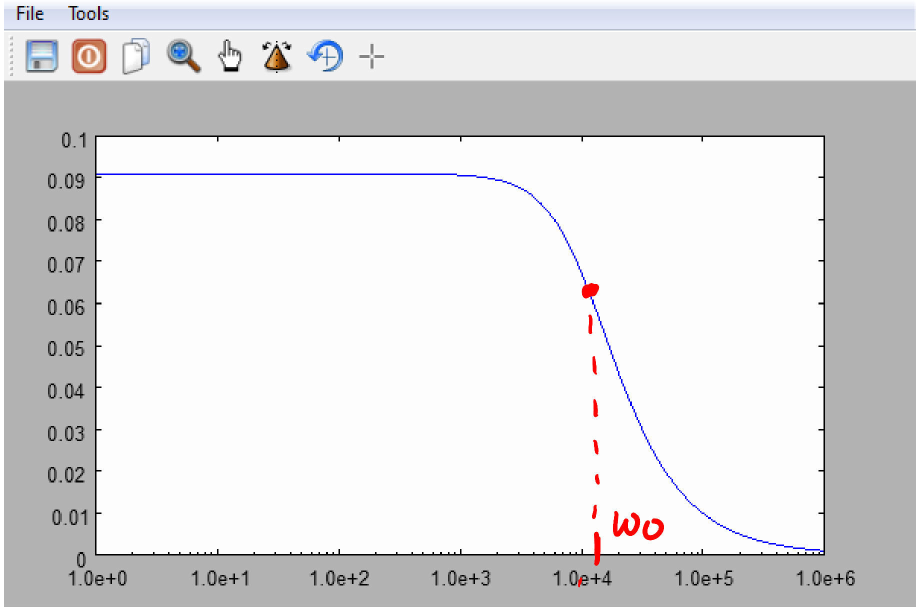
$\arg(z)$ is the complex argument »
i is the imaginary unit »

Plot:

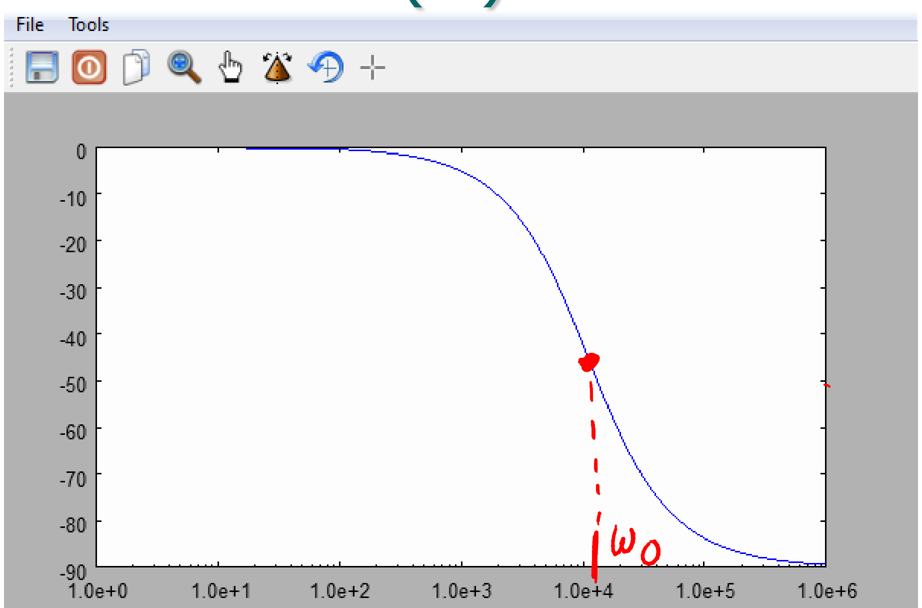


Enable interactivity

Magnitude of H(w)



Phase of H(w)



Plot in Matlab:

```
w = [1:100:1e6];
H = 1000./(j*w+11000)
plot(w, abs(H) )

semilogx(w,abs(H))        ← magnitude

semilogx(w,angle(H)*180/pi) ← phase
```

Find ω_0 for

$$H(\omega) = \frac{1000 \,/11{,}000}{(11{,}000 + j\omega)\,/11{,}000}$$

$$H(\omega) = \frac{1/11}{1 + \frac{j\omega}{11000}} = \frac{.0909}{1 + j\frac{\omega}{\omega_0}}$$

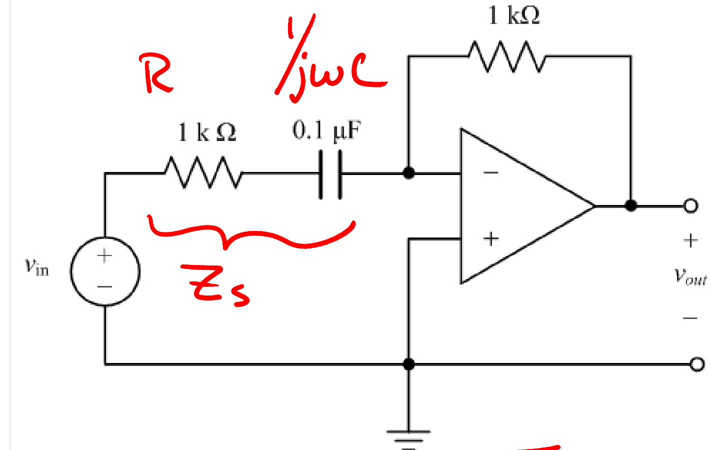
$$\omega_0 = 11{,}000$$

ω_0 is freq at which real and imaginary parts have same magnitude.

$|H(0)| = .0909 \qquad |H(\omega_0)| = \frac{.0909}{\sqrt{2}} = .064$

Ex 4) Use the Inverting Op-Amp formula to derive the transfer function. Find the cutoff frequency of the filter. Obtain the output voltage in s.s.s. when the input voltage is $V_{in}=1\cos(100t)$ V and $V_{in}=1\cos(10{,}000t+90°)$ V.

$Z_f = R$



Inverting op-Amp

$$\text{Gain} = \frac{V_{out}}{V_{in}} = H(\omega) = -\frac{Z_f}{Z_s}$$

$$H(\omega) = \frac{-R \;/R}{(R + 1/j\omega C)/R} = \frac{-1}{1 + \frac{1}{j\omega RC}}$$

$$\boxed{H(\omega) = \frac{1}{j\frac{\omega_0}{\omega} - 1}}$$

$$\omega_0 = \frac{1}{RC} = \frac{1}{(10^3)(10^{-7})} = 10^4 \text{ Rad/s}$$

at $\omega = 100$ Rad/s

$$V_{in} = 1\angle 0 \rightarrow V_{out} = V_{in} H(100) = \frac{1\angle 0}{j\left(\frac{10^4}{10^2}\right) - 1} = \frac{1}{j100 - 1} \approx \frac{-j}{100}$$

$$\therefore V_{out}(t) = .01 \cos(100t - 90°)$$

at $\omega = 10^4$ R/s

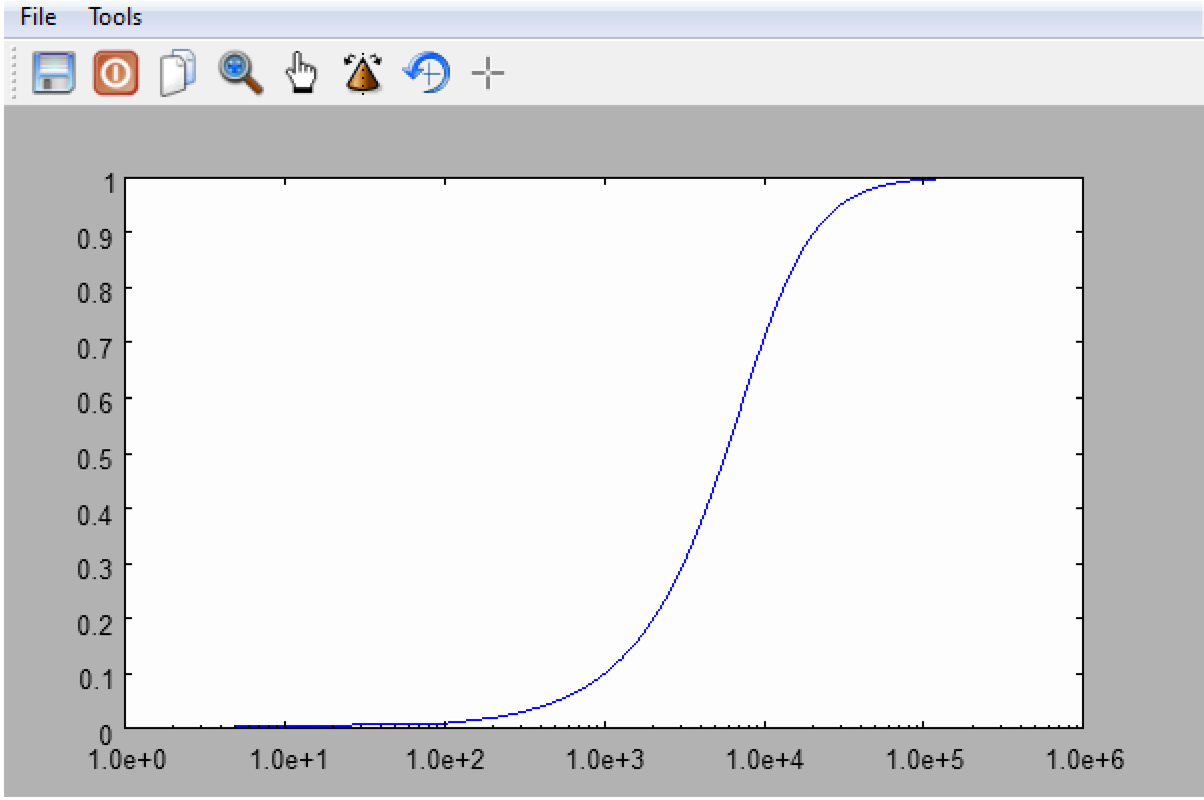
$$V_{in} = 1\angle 90° \rightarrow V_{out} = \frac{1\angle 90°}{j\left(\frac{10^4}{10^4}\right) - 1} = \frac{1\angle 90°}{\sqrt{2}\angle 135°} = .707\angle -45°$$

$$\therefore V_{out}(t) = .707 \cos(10^4 t - 45°)$$

$\omega c=10^4$ rad/s; $v_{out,\,s.s.s}=0.01\cos(100t-90°)$V, when $v_{in}=1\cos(100t)$V; $v_{out,\,s.s.s}=0.707\cos(10{,}000t-45°)$V, when $v_{in}=1\cos(10{,}000t+90°)$V

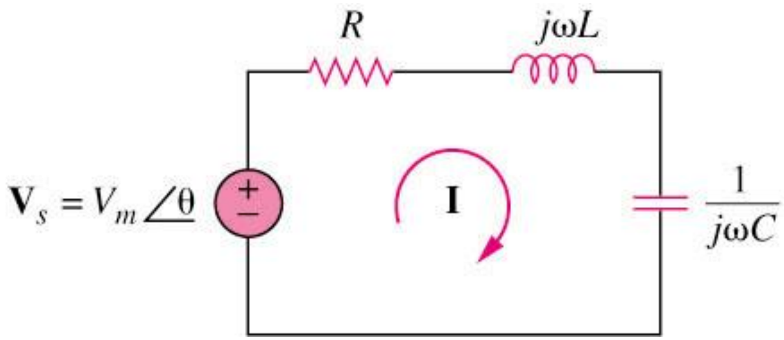
```
w = [1:100:1e6];
H = 1./ (j*(10000./w) - 1);
semilogx(w,abs(H))
```

magnitude



14.3 Series Resonance (1)

Resonance is a condition in an RLC circuit in which the capacitive and inductive reactance are equal in magnitude, thereby resulting in purely resistive impedance.



$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

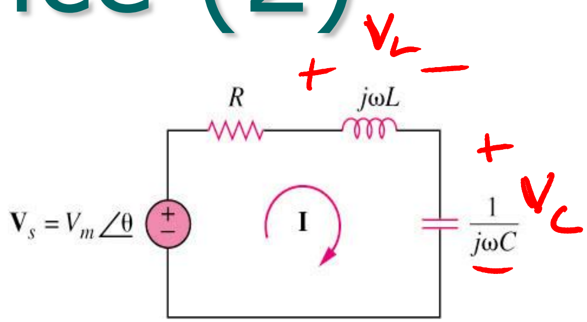
Resonance frequency:

when $\omega_0 L = \frac{1}{\omega_0 C}$

or $\omega_0 = \frac{1}{\sqrt{LC}}$

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

14.3 Series Resonance (2)



The features of series resonance:

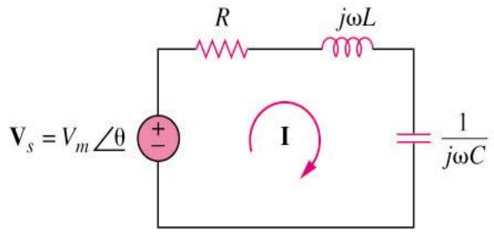
$$Z = R + j(\omega L - \frac{1}{\omega C})$$

The impedance is purely resistive, Z = R;

- The supply voltage Vs and the current I are in phase, so cos θ = 1; (power factor angle = 0)
- The magnitude of the impedance Z(ω) is minimum;
- The inductor voltage and capacitor voltage can be much more than the source voltage. 180° out of phase so cancel out

14.3 Series Resonance (3)

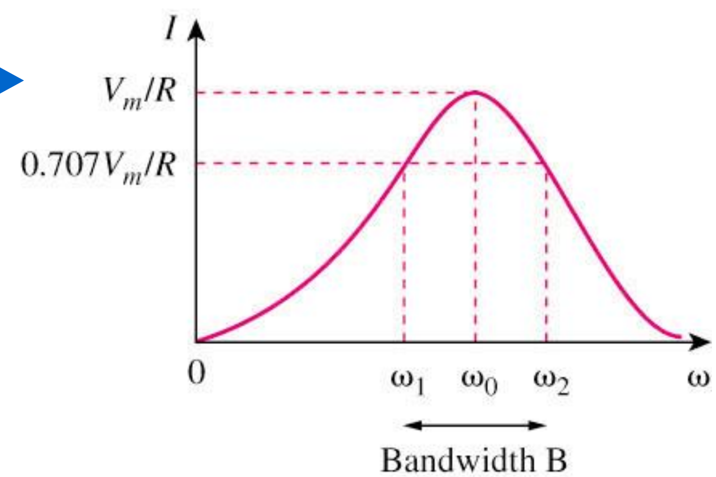
Bandwidth B



$$Z = R + j(\omega L - \frac{1}{\omega C})$$

The frequency response of the resonance circuit current is

$$I = |I| = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$



The average power absorbed by the RLC circuit is

$$P(\omega) = \frac{1}{2} I^2 R$$

The highest power dissipated occurs at resonance:

$$P(\omega_o) = \frac{1}{2} \frac{V_m^2}{R}$$

14 3 Series Resonance (4)

Half-power frequencies ω_1 and ω_2 are frequencies at which the dissipated power is half the maximum value:

$$P(\omega_1) = P(\omega_2) = \frac{1}{2}\frac{(V_m/\sqrt{2})^2}{R} = \frac{V_m^2}{4R}$$

The half-power frequencies can be obtained by setting $|Z| = R + j(\omega L - \frac{1}{\omega C})$ equal to $\sqrt{2}$ R: $\omega L - \frac{1}{\omega C} = R$, $\omega^2 - \frac{R}{L}\omega - \frac{1}{LC} = 0$

$$\omega_1 = -\frac{R}{2L} + \sqrt{(\frac{R}{2L})^2 + \frac{1}{LC}}$$

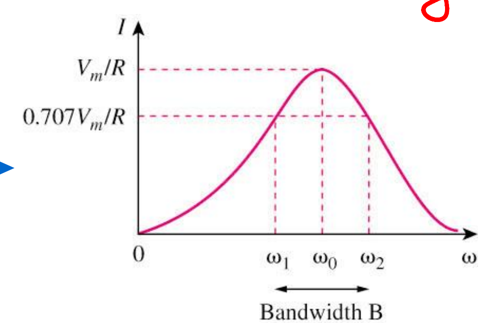
$$\omega_2 = \frac{R}{2L} + \sqrt{(\frac{R}{2L})^2 + \frac{1}{LC}}$$

$$\omega_o = \sqrt{\omega_1 \omega_2}$$

← geometric mean

Bandwidth B

$$B = \omega_2 - \omega_1$$



14.3 Series Resonance (5)

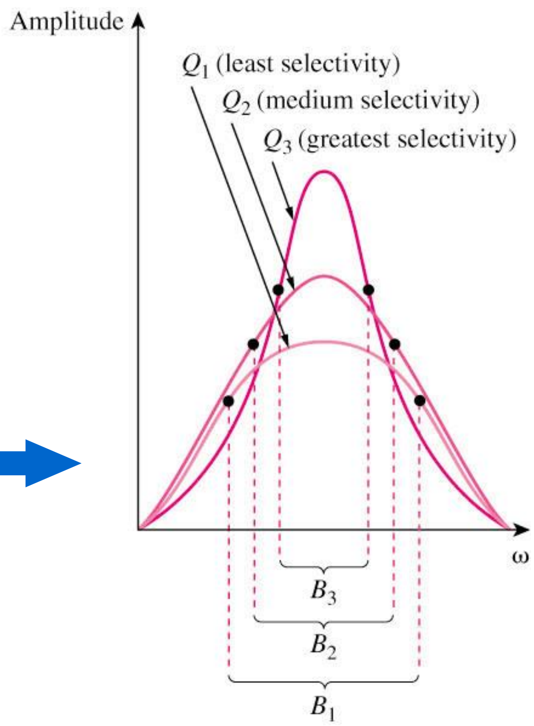
Quality factor, $Q = \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}} = \frac{\omega_o L}{R} = \frac{1}{\omega_o CR}$

The relationship between the B, Q and ω_o:

$$B = \frac{R}{L} = \frac{\omega_o}{Q} = \omega_o^2 CR$$

$Q = \frac{\omega_o}{B}$

- The quality factor is the ratio of its resonant frequency to its bandwidth.
- If the bandwidth is narrow, the quality factor of the resonant circuit must be high.
- If the band of frequencies is wide, the quality factor must be low.



Example 5

A series-connected circuit has R = 4 Ω and L = 25 mH.

a. Calculate the value of C that will produce a quality factor of 50.

b. Find ω_1 and ω_2, and B.

c. Determine the average power dissipated at $\omega = \omega_o, \omega_1, \omega_2$.

Take $V_m = 100V$.

a) $Q = \frac{\omega_o L}{R} = \frac{1}{\omega_o C R} = 50 \rightarrow \omega_o = 50\left(\frac{R}{L}\right) = 50\left(\frac{4}{.025}\right) = 8000$ R/s

then $C = \frac{1}{Q \omega_o R} = \frac{1}{(50)(8000)(4)} = .625\mu F$

b) $B = \frac{R}{L} = \frac{4}{.025} = 160$ R/s

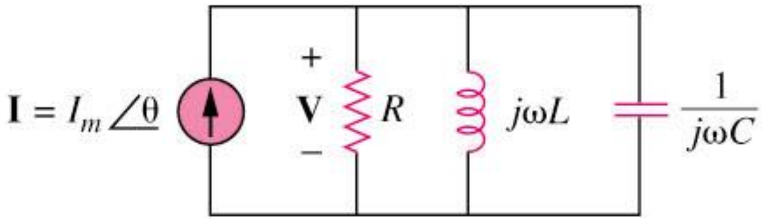
$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = -\frac{4}{.05} + \sqrt{\left(\frac{4}{.05}\right)^2 + \frac{1}{(.025)(.625\times10^{-7})}} =$

$= -80 + 8000.4 = 7920$ R/s, $\omega_2 = +80 + 8000.4 = 8080$ R/s

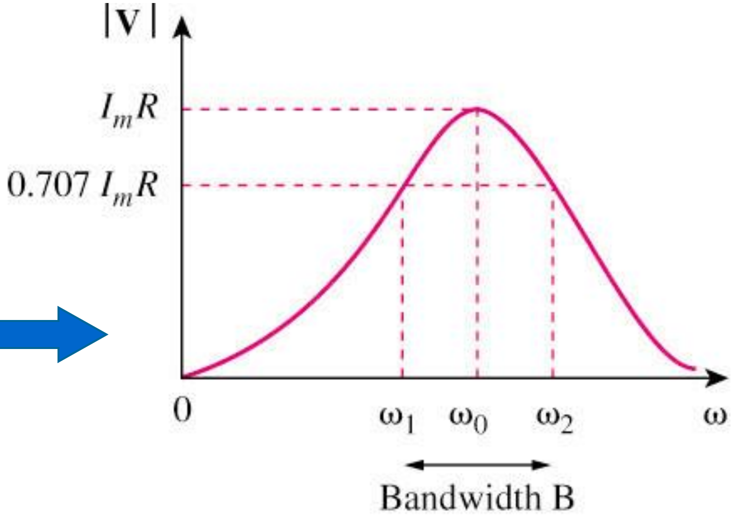
c) $P(\omega_o) = \frac{1}{2}\frac{V_m^2}{R} = \frac{1}{2}\frac{(100)^2}{4} = 1250$ W. $P(\omega_1) = P(\omega_2) = 625$ W

14.4 Parallel Resonance (1)

It occurs when imaginary part of Y is zero



$$Y = \frac{1}{R} + j(\omega C - \frac{1}{\omega L})$$



Resonance frequency:

$$\omega_o = \frac{1}{\sqrt{LC}} \text{rad/s} \quad \text{or} \quad f_o = \frac{1}{2\pi\sqrt{LC}} Hz$$

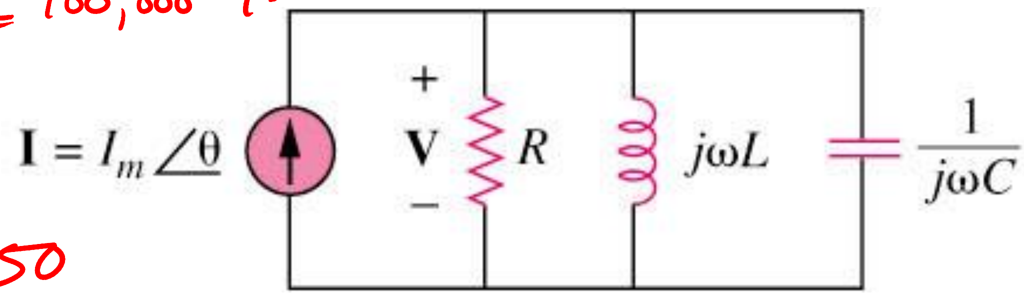
14.4 Parallel Resonance (2)

Summary of series and parallel resonance circuits:

characteristic	*Series circuit*	*Parallel circuit*
ω_o	$\frac{1}{\sqrt{LC}}$	$\frac{1}{\sqrt{LC}}$
Q	$\frac{\omega_o L}{R}$ or $\frac{1}{\omega_o RC}$	$\frac{R}{\omega_o L}$ or $\omega_o RC$
B	$\frac{\omega_o}{Q}$	$\frac{\omega_o}{Q}$
$\omega_1,\ \omega_2$	$\omega_o\sqrt{1+(\frac{1}{2Q})^2} \pm \frac{\omega_o}{2Q}$	$\omega_o\sqrt{1+(\frac{1}{2Q})^2} \pm \frac{\omega_o}{2Q}$
$Q \geq 10$, $\omega_1,\ \omega_2$	$\omega_o \pm \frac{B}{2}$	$\omega_o \pm \frac{B}{2}$

Example 6 (OPTIONAL)

A parallel resonant circuit has R = 100 kOhm, L = 20 mH, and C = 5 nF. Calculate wo, w1, w2, Q and B



$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(.02)(5\times10^{-9})}} = 100{,}000 \text{ R/s}$$

$$Q = \frac{R}{\omega_0 L} = \frac{100{,}000}{(100{,}000)(.02)} = 50$$

$$\omega_1 = \omega_0\sqrt{1+\left(\frac{1}{2Q}\right)^2} - \frac{\omega_0}{2Q} = 100{,}000\sqrt{1+\left(\frac{1}{100}\right)^2} - \frac{100000}{100}$$

$$= 100{,}000\sqrt{1.00005} - 1000$$

$$= 99{,}000 \text{ R/s}$$

$$\omega_2 = \underline{\hspace{6em}} + \frac{\omega_0}{2Q} = 101{,}000 \text{ R/s}$$

$$B = \omega_2 - \omega_1 = 2000 \text{ R/s}$$

14.5 Passive Filters (1)

- A filter is a circuit that is designed to pass signals with desired frequencies and reject or attenuate others.

- Passive filter consists of only passive element R, L and C.

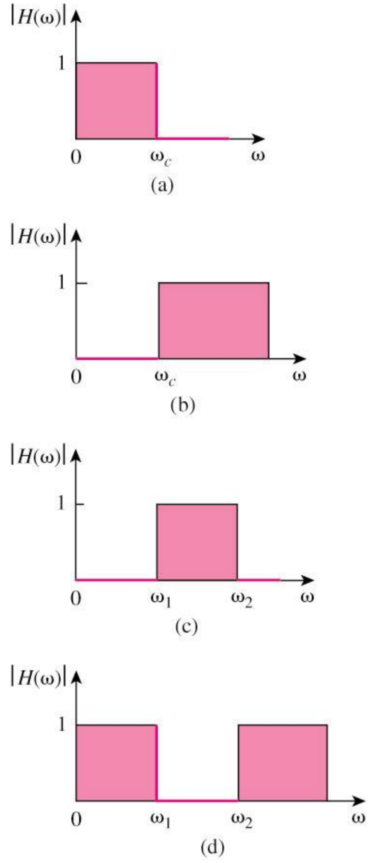
- There are four types of filters.

Low Pass

High Pass

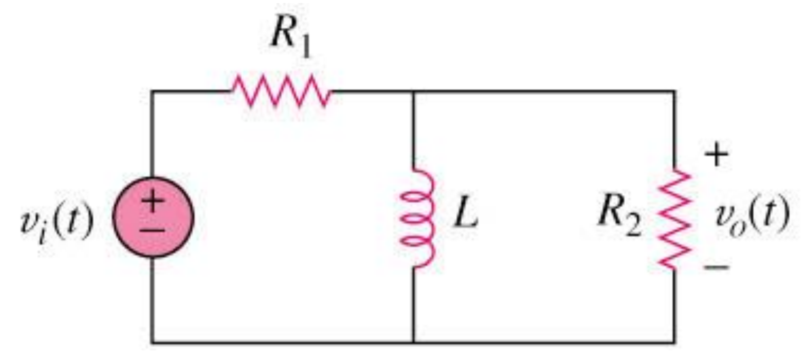
Band Pass

Band Stop



Example 7

For the circuit in the figure below, obtain the transfer function Vo(ω)/Vi(ω). Identify the type of filter the circuit represents and determine the corner frequency. Take R1=100Ω =R2 and L =2mH.



$$H(\omega) = \frac{V_o}{V_i} = \frac{Z_L \| R_2}{R_1 + Z_L \| R_2}$$

$$= \frac{\dfrac{j\omega L R_2}{j\omega L + R_2}}{\left(R_1 + \dfrac{j\omega L R_2}{j\omega L + R_2}\right)} \frac{(j\omega L + R_2)}{(j\omega L + R_2)} = \frac{j\omega L R_2 / (j\omega L)}{(j\omega L R_1 + R_1 R_2 + j\omega L R_2)/j\omega L}$$

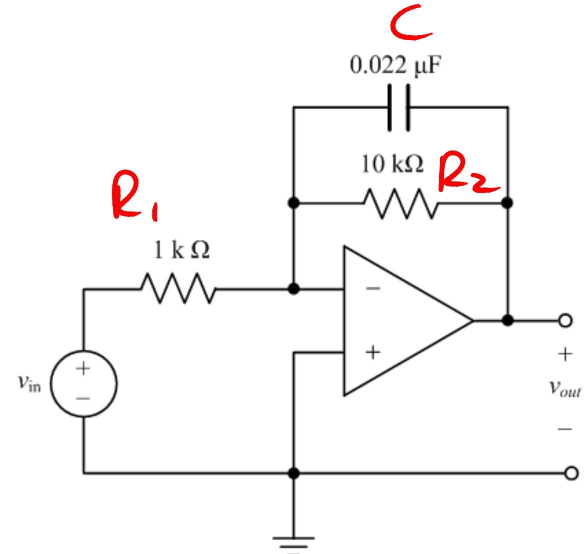
$$= \frac{R_2}{R_1 + R_2 + \dfrac{R_1 R_2}{j\omega L}} = \frac{R_2}{R_1 + R_2 - \dfrac{j R_1 R_2}{\omega L}}$$

$H(0) = 0$, $H(\infty) = \dfrac{R_2}{R_1 + R_2}$ → High Pass Filter

Corner Freq when $R_1 + R_2 = \dfrac{R_1 R_2}{\omega_c L}$ → $\omega_c = \dfrac{R_1 R_2}{(R_1 + R_2) L} = \dfrac{(100)(100)}{(200)(.002)} = 25{,}000$ r/s

Answer: $\omega = 25$ krad/s

EXAMPLE 8) a) Find the transfer function H(ω)=Vout/Vin
b) At what frequency will the magnitude of H(ω) be maximum and what is the maximum value of the magnitude of H(ω)?
c) At what frequency will the magnitude of H(ω) be minimum and what is the minimum value of the magnitude of H(ω)?



$$H(\omega)=\frac{V_{out}}{V_{in}} = -\frac{Z_F}{Z_S} = \frac{-R_2 \,||\, \frac{1}{j\omega C}}{R_1}$$

$$H(\omega) = -\frac{\left(\frac{\frac{R_2}{j\omega C}}{R_2 + 1/j\omega C}\right)}{R_1} = -\frac{1}{R_1}\left(\frac{R_2}{1+j\omega R_2 C}\right)$$

$$= \frac{-R_2}{R_1 + j\omega R_1 R_2 C} = \frac{-R_2/R_1 R_2 C}{j\omega + \frac{R_1}{R_1 R_2 C}} = \frac{-\frac{1}{(1000)(.022\times10^{-6})}}{j\omega + \frac{1}{(10000)(.022\times10^{-6})}}$$

$$= \frac{-45455}{j\omega + 4545.5}$$

$H_{min} = 0$ when $\omega \to \infty$

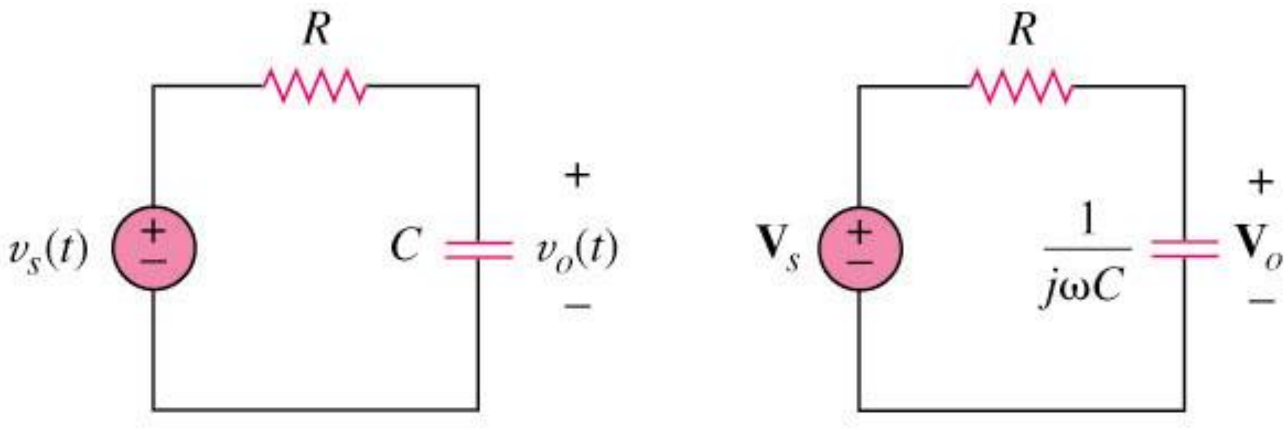
$H_{max} = 10$ when $\omega \to 0$

a) H(ω)= -45455/(jω+4545.5)
b) Hmax=10 as ω=0 rad/s
c) Hmin=0 as ω goes to infinity.

HANDOUTS

Example 1

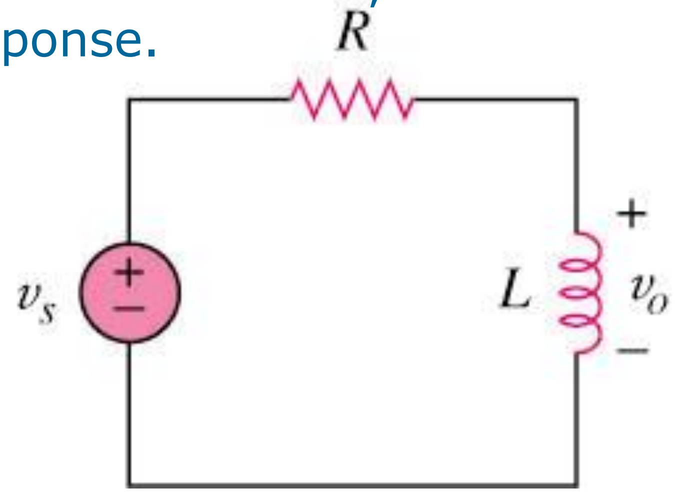
For the RC circuit shown below, obtain the transfer function Vo/Vs and its frequency response. Let $v_s = V_m \cos\omega t$.



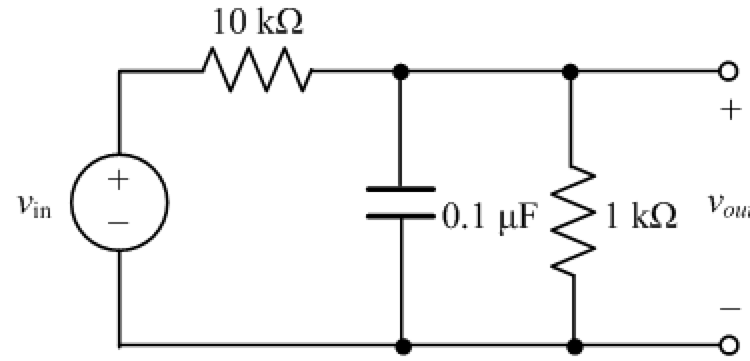
(a) (b)

Example 2

Obtain the transfer function Vo/Vs of the RL circuit shown below, assuming $v_s = V_m \cos\omega t$. Sketch its frequency response.

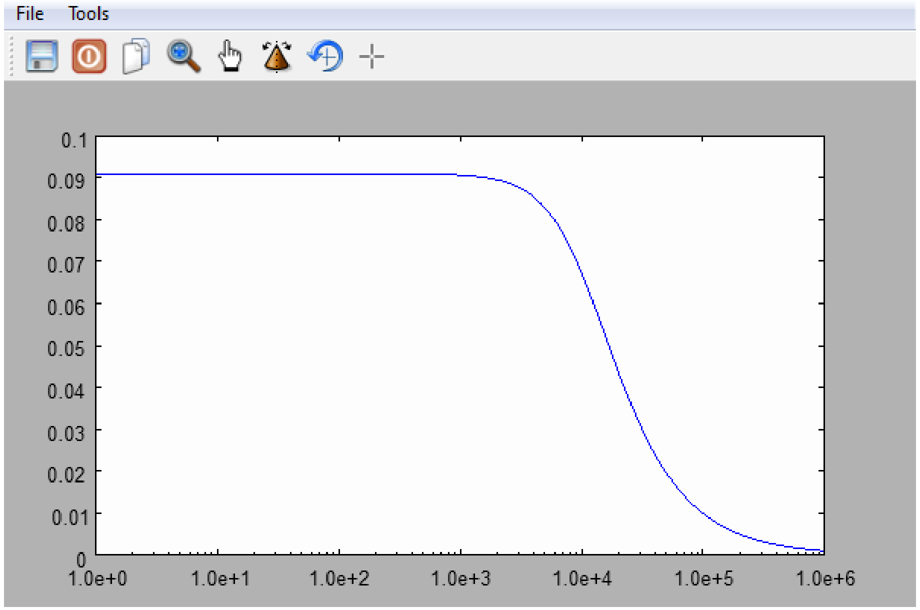


Ex3) Find the transfer function $H(\omega)=V_{out}/V_{in}$. What type of filter is it?



H(w)=1000/(jw+11000), Lowpass filter

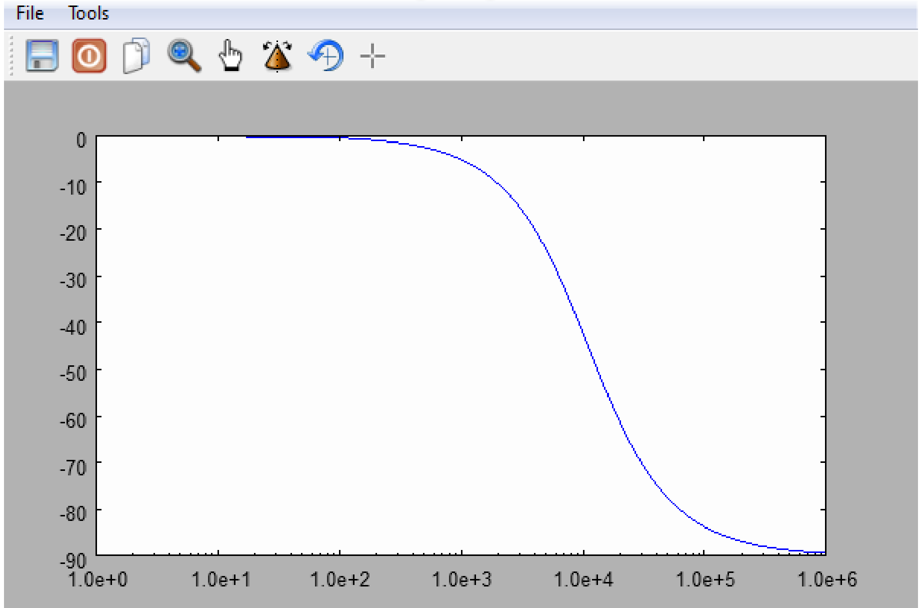
Magnitude of H(w)



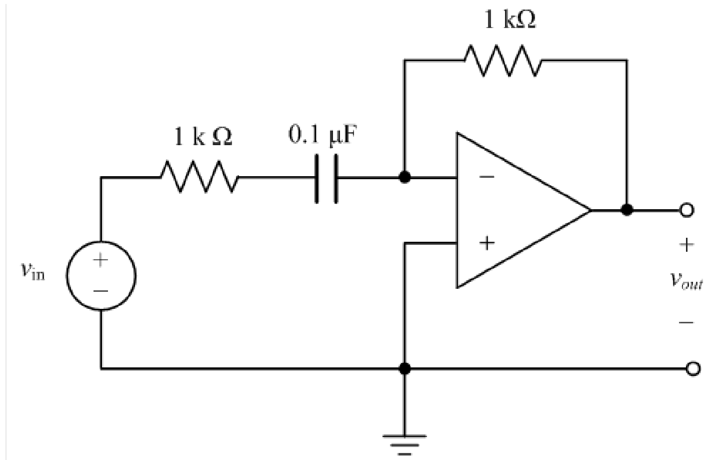
Plot in Matlab:

```
w = [1:100:1e6];
H = 1000./(j*w+11000)
plot(w, abs(H) )

semilogx(w,abs(H))        ← magnitude

semilogx(w,angle(H)*180/pi) ← phase
```

Phase of H(w)



Ex 4)Find the cutoff frequency of the filter. Use voltage divider to derive the transfer function. Obtain the output voltage in s.s.s when the input voltage is V_{in}=1cos(100t) V and V_{in}=1cos(10,000t+90°) V.



ωc=10^4 rad/s; $v_{out,\ s.s.s}$=0.01cos(100t-90°)V, when v_{in}=1cos(100t)V; $v_{out,\ s.s.s}$=0.707cos(10,000t-45°)V, when v_{in}=1cos(10,000t+90°)V

Example 5

A series-connected circuit has $R = 4\ \Omega$ and $L = 25$ mH.

a. Calculate the value of C that will produce a quality factor of 50.

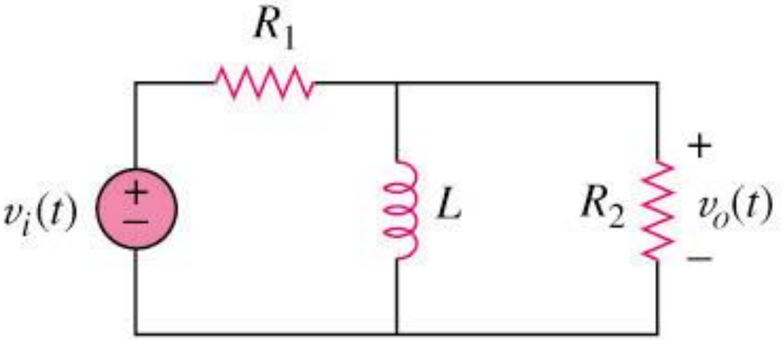
b. Find ω_1 and ω_2, and B.

c. Determine the average power dissipated at $\omega = \omega_o, \omega_1, \omega_2$.

Take $V_m = 100$V.

Example 7

For the circuit in the figure below, obtain the transfer function Vo(ω)/Vi(ω). Identify the type of filter the circuit represents and determine the corner frequency. Take R1=100Ω =R2 and L =2mH.



Answer: $\omega = 25\ \text{krad/s}$

EXAMPLE 8) a) Find the transfer function H(ω)=Vout/Vin
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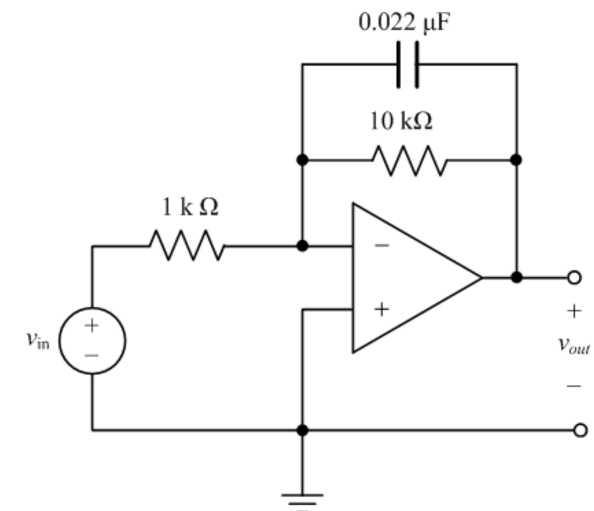

a) H(ω)= -45455/(jω+4545.5)
b) Hmax=10 as ω=0 rad/s
c) Hmin=0 as ω goes to infinity.